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# QUANTUM STOCHASTIC SEMIGROUPS AND THEIR GENERATORS

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ABSTRACT. A rigged Hilbert space characterisation of the unbounded generators of quantum completely positive (CP) stochastic semigroups is given. The general form and the dilation of the stochastic completely dissipative (CD) equation over the algebra $\mathcal{L}(\mathcal{H})$ is described, as well as the unitary quantum stochastic dilation of the subfiltering and contractive flows with unbounded generators is constructed.

## INTRODUCTION

Quantum stochastic dynamics gives beautiful solvable models for the interaction of a quantum system with the quantum noise, which is produced by a heat bath, measurement apparatus, or any other environment with infinite number of freedom. It can be defined by a weakly continuous evolution semigroup on a rigged Hilbert space with a special unbounded form-generator, corresponding to the singular boundary-type interaction. The Heisenberg picture of such interaction is described by the quantum stochastic Langevin equation.

In quantum theory of open systems there is a well known Lindblad's form [1] of quantum Markovian master equation, satisfied by the one-parameter semigroup of completely positive (CP) maps over the algebra $\mathcal{L}(\mathcal{H})$ of bounded operators on the system Hilbert space $\mathcal{H}$. This is nonstochastical equation, which can be obtained by averaging stochastic Langevin equation for quantum flow [2] over the driving noises, represented in a Fock space $\mathcal{F}$. On the other hand the quantum EH-flow corresponds to the interaction representation for a one parametric group of dynamical authomorphisms over $\mathcal{L}(\mathcal{H}\otimes\mathcal{F})$, which are obviously completely positive due to *-multiplicativity of these representations. The authomorphisms (representations) give the examples of pure, i.e. extreme point CP maps, but among the extreme points of the convex cone of all CP maps over $\mathcal{L}(\mathcal{H})$ there are not only the representations. This means a possibility to construct the stochastic representations of dynamical CP semigroups as averagings of pure, i.e. non-mixing irreversible quantum stochastic CP dynamics, which can not be driven by a Langevin equation. Such irreversible dynamics, corresponding to the interaction representation for a dynamical CP semigroup over $\mathcal{L}(\mathcal{H}\otimes\mathcal{F})$, are described by quantum stochastic flows of CP maps, which should satisfy a generalized form of Lindblad equation with quantum stochastic unbounded generators.

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The examples of such dynamics having recently been found many physical applications, will be considered in the first section. The rest of the paper will be devoted to the mathematical derivation of the general structure for the unbounded generators of the dynamical CP semigroups, corresponding to the quantum stochastic CP flows over $\mathcal{L}(\mathcal{H})$ with the noises represented in $\mathcal{F}$. The results of the paper not only generalize the Evans-Hudson (EH) flows [2] from the representations to the general CP maps, but also prove the existence of the homomorphic dilations for the subfiltering and contractive CP flows. This gives the subfiltering CP flows as conditional expectations of EH flows, generalizing the similar representation for contractive CP semigroups. Here in the introduction we would like to outline the generalized structure of the generators on the formal level.

As was proved in [3], every stationary quantum stochastic processes $t \in \mathbb{R}_+ \mapsto \Lambda(t,a)$ parametrized by $a \in \mathfrak{a}$ with $\Lambda(0,a) = 0$ and independent increments $\mathrm{d}\Lambda(t,a) = \Lambda(t+\mathrm{d}t,a) - \Lambda(t,a)$, forming an Itô $\star$-algebra

(0.1)
$$\mathrm{d}\Lambda(a)^* \mathrm{d}\Lambda(a) = \mathrm{d}\Lambda(a^\star a), \quad \sum \lambda_i \mathrm{d}\Lambda(a_i) = \mathrm{d}\Lambda\left(\sum \lambda_i a_i\right), \quad \mathrm{d}\Lambda(a)^* = \mathrm{d}\Lambda(a^\star),$$

can be represented in the Fock space $\mathfrak{F}$ over the space of $\mathcal{E}$ -valued square-integrable functions on $\mathbb{R}_+$ as $\Lambda(t,a) = a^\mu_\nu \Lambda^\nu_\mu(t)$ with respect to the vacuum state $\delta_\emptyset \in \mathfrak{F}$. Here

$$a^\mu_\nu \Lambda^\nu_\mu(t) = a^\bullet_\bullet \Lambda^\bullet_\bullet(t) + a^\bullet_+ \Lambda^+_\bullet(t) + a^-_\bullet \Lambda^\bullet_-(t) + a^-_+ \Lambda^+_-(t), \tag{0.2}$$

is the canonical decomposition of $\Lambda$ into the exchange $\Lambda^\bullet_\bullet$, creation $\Lambda^+_\bullet$, annihilation $\Lambda^\bullet_-$ and preservation (time) $\Lambda^+_- = t\mathrm{I}$ processes of quantum stochastic calculus [4], [5] having the mean values $\left\langle \Lambda^\nu_\mu(t) \right\rangle = t\delta^\nu_+ \delta^-_\mu$ with respect to the vacuum state in $\mathfrak{F}$, and $\mathcal{E}$ is a pre-Hilbert space of the quantum noise in $\mathfrak{F}$. Thus the parametrizing algebra $\mathfrak{a}$ can be always identified with a $\star$-subalgebra of the algebra $\mathcal{Q}(\mathcal{E})$ of all quadruples $\boldsymbol{a} = (a^\mu_\nu)^{\mu=-,\bullet}_{\nu=+,\bullet}$, where $a^\mu_\nu : \mathcal{E}_\nu \to \mathcal{E}_\mu$ are the linear operators on $\mathcal{E}_\bullet = \mathcal{E}, \mathcal{E}_+ = \mathbb{C} = \mathcal{E}_-$, having the adjoints $a^{\mu *}_\nu \mathcal{E}_\mu \subseteq \mathcal{E}_\nu$, with the Hudson–Parthasarathy (HP) multiplication table [6]

$$\boldsymbol{a} \bullet \boldsymbol{b} = (a^\mu_\bullet b^\bullet_\nu)^{\mu=-;\bullet}_{\nu=+,\bullet}, \tag{0.3}$$

and the involution $a^{\star\mu}_{-\nu} = a^{\nu *}_{-\mu}$, where $-(-) = +$, $-\bullet = \bullet$, $-(+) = -$.

The stochastic differential of a CP flow $\phi = (\phi_t)_{t>0}$ over an operator algebra $\mathcal{B} \subseteq \mathcal{L}(\mathcal{H})$ is written in terms of the quantum canonical differentials as $\mathrm{d}\phi = \phi \circ \lambda^\mu_\nu \mathrm{d}\Lambda^\nu_\mu$ with $\phi_0 = \imath$ at $t = 0$, where $\imath(B) = B$ is the identical representation of $\mathcal{B}$. The main result of this paper is the derivation and the dilation of the linear quantum stochastic evolution equation

$$\mathrm{d}\phi_t(B) + \phi_t(K^*B + BK - L^* \jmath(B) L)\,\mathrm{d}t = \phi_t(L^\bullet \jmath(B) L_\bullet - B \otimes \delta^\bullet_\bullet)\,\mathrm{d}\Lambda^\bullet_\bullet$$

$$+\phi_t(L^\bullet \jmath(B) L - K^\bullet B)\,\mathrm{d}\Lambda^+_\bullet + \phi_t(L^* \jmath(B) L_\bullet - BK_\bullet)\,\mathrm{d}\Lambda^\bullet_-, \tag{0.4}$$

where $\jmath$ is an operator representation of $\mathcal{B}$, $\delta^\bullet_\bullet$ is the identity operator in $\mathcal{E}$, and the operator $K$ satisfies the dissipativity condition $K + K^\dagger \geq L^*L$ with the Hamiltonian part $H = \operatorname{Im} K$. This differential form for the CP flows was discovered in [7] as the general completely dissipative (CD) structure of the bounded quantum stochastic generators $\lambda^\mu_\nu : \mathcal{B} \to \mathcal{B}$ over a von Neumann algebra $\mathcal{B}$ even in the nonlinear case.

In the matrix form $\boldsymbol{\lambda} = (\lambda^\mu_\nu)^{\mu=-,\bullet}_{\nu=+,\bullet}$ this can be written similar to the Lindblad form for the nonstochastic generator $\lambda = \lambda^-_+$ as

$$\boldsymbol{\lambda}(B) = \boldsymbol{L}^* \jmath(B)\boldsymbol{L} - \boldsymbol{K}^* B - B\boldsymbol{K}. \tag{0.5}$$

The dilation of the stochastic differentials for CP processes over arbitrary $*$-algebras $\mathfrak{a}$ and $\mathcal{B}$, giving this structure for the bounded generators over a von Neumann algebra $\mathcal{B}$ as a consequence of the Christensen-Evans theorem [8], was obtained in [7, 9].

Here we shall prove that the quantum stochastic extension (0.5) of the Lindblad's structure $\lambda(B) = L^* \jmath(B) L - K^* B - BK$, can always be used for the construction and the dilation of the CP flows also in the case of the unbounded maps $\lambda^\mu_\nu$ over the algebra $\mathcal{B} = \mathcal{L}(\mathcal{H})$. The existence of minimal CP solution which has been recently constructed under certain continuity conditions in [10] proves that this structure is also sufficient for the CP property of any solution to this stochastic equation. The construction of the differential dilations and the CP solutions of such quantum stochastic differential equations with the bounded generators over the simple finite-dimensional Itô algebra $\mathfrak{a} = \mathcal{Q}(\mathcal{E})$ and the arbitrary $\mathcal{B} \subseteq \mathcal{L}(\mathcal{H})$ was also recently discussed in [11, 12].

The nonstochastic case $\Lambda(t, a) = \alpha t \mathrm{I}$ is described by the simplest, one-dimensional Itô algebra $\mathfrak{a} = \mathbb{C}d$ with $l(a) = \alpha \in \mathbb{C}$ and the nilpotent multiplication $\alpha^\star \alpha = 0$ corresponding to the non-stochastic (Newton) calculus $(\mathrm{d}t)^2 = 0$ in $\mathcal{E} = 0$. The standard Wiener process $\mathrm{Q} = \Lambda^\bullet_- + \Lambda^+_\bullet$ in Fock space is described by the second order nilpotent algebra $\mathfrak{a}$ of pairs $a = (\alpha, \xi)$ with $d = (1, 0)$, $\xi \in \mathbb{C}$, represented by the quadruples $a^-_+ = \alpha$, $a^-_\bullet = \xi = a^\bullet_+$, $a^\bullet_\bullet = 0$ in $\mathcal{E} = \mathbb{C}$, corresponding to $\Lambda(t, a) = \alpha t \mathrm{I} + \xi \mathrm{Q}(t)$. The unital $\star$-algebra $\mathbb{C}$ with the usual multiplication $\zeta^\star \zeta = |\zeta|^2$ can be embedded into the two-dimensional Itô algebra $\mathfrak{a}$ of $a = (\alpha, \zeta)$, $\alpha = l(a)$, $\zeta \in \mathbb{C}$ as $a^\bullet_\bullet = \zeta$, $a^\bullet_+ = +i\zeta$, $a^-_\bullet = -i\zeta$, $a^-_+ = \zeta$. It corresponds to $\Lambda(t, a) = \alpha t \mathrm{I} + \zeta \mathrm{P}(t)$, where $\mathrm{P} = \Lambda^\bullet_\bullet + i\left(\Lambda^+_\bullet - \Lambda^\bullet_-\right)$ is the representation of the standard Poisson process, compensated by its mean value $t$. These two commutative cases exhaust the possible types of two-dimensional Itô algebras. Thus, our results [9, 10] are also applicable to the classical stochastic differentials of completely positive processes, corresponding to the commutative Itô algebras, which are always decomposable into the Wiener, Poisson and Newton orthogonal components.

## 1. Quantum sub-filtering dynamics

The quantum filtering theory, which was outlined in [13, 14] and developed then since [15], provides the derivations for new types of irreversible stochastic equations for quantum states, giving the dynamical solution for the well-known quantum measurement problem. Some particular types of such equations have been considered also in the phenomenological theories of quantum permanent reduction [16, 17], continuous measurement collapse [18, 19], spontaneous jumps [26, 20], diffusions and localizations [21, 22]. The main feature of such dynamics is that the reduced irreversible evolution can be described in terms of a linear dissipative stochastic wave equation, the solution to which is normalized only in the mean square sense.

The simplest dynamics of this kind is described by the continuous filtering wave propagators $V_t(\omega)$, defined on the space $\Omega$ of all Brownian trajectories as an adapted operator-valued stochastic process in the system Hilbert space $\mathcal{H}$, satisfying the

stochastic diffusion equation

$$\mathrm{d}V_t + KV_t\mathrm{d}t = LV_t\mathrm{dQ}, \quad V_0 = I \tag{1.1}$$

in the Itô sense. Here $\mathrm{Q}(t,\omega)$ is the standard Wiener process, which is described by the independent increments $\mathrm{dQ}(t) = \mathrm{Q}(t+\mathrm{d}t) - \mathrm{Q}(t)$, having the zero mean values $\langle \mathrm{dQ}\rangle = 0$ and the multiplication property $(\mathrm{dQ})^2 = \mathrm{d}t$, $K$ is an accretive operator, $K + K^\dagger \geq L^* L$, defined on a dense domain $\mathcal{D} \subseteq \mathcal{H}$, with $K^\dagger = K^*|\mathcal{D}$, and $L$ is a linear operator $\mathcal{D} \to \mathcal{H}$. This stochastic wave equation with $K + K^\dagger = L^* L$ was first derived [24] from a unitary cocycle evolution by a quantum filtering procedure. A sufficient analyticity condition, under which it has the unique solution in the form of stochastic multiple integral even in the case of unbounded $K$ and $L$ is given in [10]. Using the Itô formula

$$\mathrm{d}(V_t^* V_t) = \mathrm{d}V_t^* V_t + V_t^* \mathrm{d}V_t + \mathrm{d}V_t^* \mathrm{d}V_t, \tag{1.2}$$

and averaging $\langle\cdot\rangle$ over the trajectories of Q, one obtains $\langle V_t^* V_t\rangle \leq I$ as a consequence of $\mathrm{d}\langle V_t^* V_t\rangle \leq 0$. Note that the process $V_t$ is not necessarily unitary if the filtering condition $K^\dagger + K = L^* L$ holds, and even if $L^\dagger = -L$, it might be only isometric, $V_t^* V_t = I$, in the unbounded case.

Another type of the filtering wave propagator $V_t(\omega) : \psi_0 \in \mathcal{H} \mapsto \psi_t(\omega)$ in $\mathcal{H}$ is given by the stochastic jump equation

$$\mathrm{d}V_t + KV_t\mathrm{d}t = LV_t\mathrm{dP}, \quad V_0 = I. \tag{1.3}$$

at the random time instants $\omega = \{t_1, t_2, ...\}$. Here $L = J - I$ is the jump operator, corresponding to the stationary discontinuous evolutions $\psi_{t+} = J\psi$ at $t \in \omega$, and $\mathrm{P}(t,\omega) = \mathrm{N}(t,\omega) - t$ is the standard Poisson process, counting the number $\mathrm{N}(t,\omega) = |\omega \cap [0,t)|$ compensated by its mean value $t$. It is described as the process with independent increments $\mathrm{dP}(t) = \mathrm{P}(t+\mathrm{d}t) - \mathrm{P}(t)$, having the values $\{0,1\}$ at $\mathrm{d}t \to 0$, with zero mean $\langle \mathrm{dP}\rangle = 0$, and the multiplication property $(\mathrm{dP})^2 = \mathrm{dP} + \mathrm{d}t$. This stochastic wave equation was first derived in [23] under the filtering condition $L^* L = K + K^\dagger$ by the conditioning with respect to the spontaneous reductions $J : \psi_t \mapsto \psi_{t+}$. An analyticity condition under which it has the unique solution in the form of the multiple stochastic integral even in the case of unbounded $K$ and $L$ is also given in [10]. Using the Itô formula (1.2) with $\mathrm{d}V_t^* \mathrm{d}V_t = V_t^* L^* L V_t(\mathrm{dP}+\mathrm{d}t)$, one can obtain

$$\mathrm{d}(V_t^* V_t) = V_t^* \left(L^* L - K - K^\dagger\right) V_t \mathrm{d}t + V_t^* \left(L^\dagger + L + L^* L\right) V_t \mathrm{dP}.$$

Averaging $\langle\cdot\rangle$ over the trajectories of P, one can easily find that $\mathrm{d}\langle V_t^* V_t\rangle \leq 0$ under the sub-filtering condition $L^* L \leq K + K^\dagger$. Such evolution is not needed to be unitary, but in the filtering case it might be isometric, $V_t^* V_t = I$ if the jumps are isometric, $J^* J = I$.

This proves in both cases that the stochastic wave function $\psi_t(\omega) = V_t(\omega)\psi_0$ is not normalized for each $\omega$, but it is normalized in the mean square sense to the survival probability $\langle ||\psi_t||^2\rangle \leq ||\psi_0||^2 = 1$, a decreasing probability for a quantum unstable system not to be demolished during its observation up to the time $t$. In the stable case $\left\langle ||\psi_t||^2\right\rangle = 1$ the positive stochastic function $p_t(\omega) = ||\psi_t(\omega)||^2$ is the probability density of a diffusive $\widehat{\mathrm{Q}}$ or counting $\widehat{\mathrm{P}}$ output process up to the given $t$ with respect to the standard Wiener Q or Poisson P input processes correspondingly, in the general case this is given by the conditional probability density $||\psi_t(\omega)||^2 / \langle ||\psi_t||^2\rangle$.

Using the Itô formula for $\rho_t(\omega) = V_t(\omega)\rho_0 V_t(\omega)^*$, one can obtain the stochastic equations

$$\mathrm{d}\rho_t + (K\rho_t + \rho_t K^* - L\rho_t L^*)\,\mathrm{d}t = (L\rho_t + \rho_t L^*)\,\mathrm{dQ}, \tag{1.4}$$

$$\mathrm{d}\rho_t + (K\rho_t + \rho_t K^* - L\rho_t L^*)\,\mathrm{d}t = (J\rho_t J^* - \rho_t)\,\mathrm{dP}, \tag{1.5}$$

describing the stochastic evolution $\Phi_t : \rho_0 \mapsto \rho_t$ of an initially normalized density operator $\rho_0 \geq 0$, $\mathrm{tr}\rho_0 = 1$ as the stochastic density operator $\rho_t(\omega) = \Phi_t(\omega, \rho_0)$, normalized to the probability density $p_t(\omega) = \mathrm{tr}\rho_t(\omega)$. The stochastic dynamical maps $\Phi_t(\rho) = V_t\rho V_t^*$ are obviously positive but in general irreversible if $V_t(\omega)$ are not unitary, although they preserve the pure states in this particular case.

Although the filtering equations (1.3), (1.1) look very different, they can be unified in the form of quantum stochastic equation

$$\mathrm{d}V_t + KV_t\mathrm{d}t + K^- V_t \mathrm{d}\Lambda_- = (J - I)\,V_t\mathrm{d}\Lambda + L_+ V_t \mathrm{d}\Lambda^+ \tag{1.6}$$

where $\Lambda^+(t)$ is the creation process, corresponding to the annihilation $\Lambda_-(t)$ on the interval $[0, t)$, and $\Lambda(t)$ is the number of quanta on this interval. Indeed, the standard Poisson process P as well as the Wiener process Q can be represented in $\mathfrak{F}$ by the linear combinations [6]

$$\mathrm{P}(t) = \Lambda(t) + i\left(\Lambda^+(t) - \Lambda_-(t)\right), \quad \mathrm{Q}(t) = \Lambda^+(t) + \Lambda_-(t), \tag{1.7}$$

so the equation (1.6) corresponds to the stochastic diffusion equation (1.1) if $J = I$, $L_+ = L = -K^-$, and it corresponds to the stochastic jump equation (1.3) if $J = I + L$, $L_+ = iL = K^-$. These canonical quantum stochastic processes, representing the quantum noise with respect to the vacuum state $|0\rangle$ of the Fock space $\mathcal{F}$ over the single-quantum Hilbert space $L^2(\mathbb{R}_+)$ of square-integrable functions of $t \in [0, \infty)$, are formally given in [25] by the integrals

$$\Lambda_-(t) = \int_0^t \Lambda_-^r \mathrm{d}r, \quad \Lambda^+(t) = \int_0^t \Lambda_r^+ \mathrm{d}r, \quad \Lambda(t) = \int_0^t \Lambda_r^+ \Lambda_-^r \mathrm{d}r,$$

where $\Lambda_-^r, \Lambda_r^+$ are the generalized quantum one-dimensional fields in $\mathcal{F}$, satisfying the canonical commutation relations

$$\left[\Lambda_-^r, \Lambda_s^+\right] = \delta(s - r)\,I, \quad \left[\Lambda_-^r, \Lambda_-^s\right] = 0 = \left[\Lambda_r^+, \Lambda_s^+\right].$$

They can be defined by the independent increments with

$$\langle 0|\mathrm{d}\Lambda_-|0\rangle = 0, \quad \langle 0|\mathrm{d}\Lambda^+|0\rangle = 0, \quad \langle 0|\mathrm{d}\Lambda|0\rangle = 0 \tag{1.8}$$

and the noncommutative multiplication table

$$\mathrm{d}\Lambda\mathrm{d}\Lambda = \mathrm{d}\Lambda, \quad \mathrm{d}\Lambda_-\mathrm{d}\Lambda = \mathrm{d}\Lambda_-, \quad \mathrm{d}\Lambda\mathrm{d}\Lambda^+ = \mathrm{d}\Lambda^+, \quad \mathrm{d}\Lambda_-\mathrm{d}\Lambda^+ = \mathrm{d}tI \tag{1.9}$$

with all other products being zero: $\mathrm{d}\Lambda\mathrm{d}\Lambda_- = \mathrm{d}\Lambda^+\mathrm{d}\Lambda = \mathrm{d}\Lambda^+\mathrm{d}\Lambda_- = 0$.

The corresponding quantum stochastic equation for the density operator $\rho_t = V_t\rho_0 V_t^*$ has the following form

$$\mathrm{d}\rho_t + (K\rho_t + \rho_t K^* - L\rho_t L^*)\,\mathrm{d}t = (J\rho_t J^* - \rho_t)\,\mathrm{d}\Lambda$$

$$+ \left(J\rho_t L^- - K^-\rho_t\right)\mathrm{d}\Lambda_- + (L_+\rho_t J^* - \rho_t K_+)\,\mathrm{d}\Lambda^+, \tag{1.10}$$

where $L^{-} = L_{+}^{*}, K_{+}^{*} = K^{-}$. The equation (1.10), coinciding with either (1.4) or with (1.5) in the particular cases, is obtained from (1.6) by using the Itô formula (1.2) with the multiplication table (1.9). In the another particular case

$$J = S, \quad K^{-} = L^{-}S, \quad L_{+} = SK_{+}, \quad S^{*}S = I,$$

it corresponds to the Hudson–Evans quantum stochastic flow [2] if $S^{*} = S^{-1}$. Such evolution is isometric, and identity preserving, $V_t V_t^{*} = I$, i.e. unitary at least in the case of the bounded $K$ and $L$.

In the Heisenberg picture the stochastic dynamics is described by the dual transformations $\phi_t(\omega) = \Phi_t'(\omega)$, such that for any density operator $\rho_0$ and for any bounded observable $B$ on $\mathcal{H}$

$$\operatorname{tr}\left[\Phi_t'(\omega, B)\,\rho_0\right] = \operatorname{tr}\left[B\Phi_t(\omega, \rho_0)\right].$$

The linear stochastic maps $B \mapsto Y_t = \phi_t(B)$ are obviously Hermitian in the sense that $Y_t^{*} = Y_t$ if $B^{*} = B$ and completely positive, but in contrast to the usual Hamiltonian dynamics, they are multiplicative, $\phi_t(B^{*}C) = \phi_t(B)^{*}\phi_t(C)$ only in the case, corresponding to the HE flow, even if they are not averaged with respect to $\omega$. Moreover, they are usually not normalized, $R_t(\omega) := \phi_t(\omega, I) \neq I$, although the stochastic positive operators $R_t = V_t^{*}V_t$ under the filtering condition are usually normalized in the mean, $\langle R_t \rangle = I$, and satisfy the martingale property $\epsilon_t[R_s] = R_t$ for all $s > t$, where $\epsilon_t$ is the conditional expectation with respect to the history of the processes P or Q up to time $t$. The sub-filtering condition $K + K^{\dagger} \geq L^{-}L_{+}$ for the equation (1.6) defines in both cases the positive operator-valued stochastic process $R_t = \phi_t(I)$ as a sub-martingale with $R_0 = I$, or a martingale in the case $K + K^{\dagger} = L^{-}L_{+}$.

Although the filtering dynamics with unbounded coefficients of the particular types has been studied elsewhere [27] by means of the classical stochastic differential equations, the general structure of such equations has not been discovered, and the general filtering CP flows have not been constructed. In the next sections we define a multidimensional analog of the quantum stochastic equation (1.10), and will show that the general structure of its generator indeed follows just from the property of complete positivity of the dual stochastic maps $\phi_t = \Phi_t'$ for all $t > 0$ and the normalization condition $\phi_t(I) = R_t$ to a form-valued sub-martingale with respect to the natural filtration of the quantum noise in the Fock space $\mathfrak{F}$ .

## 2. Quantum completely positive flows

Throughout the complex pre-Hilbert space $\mathcal{D} \subseteq \mathcal{H}$ is a Fréchet (i.e. metrizable complete) space with respect to a stronger topology, $\mathcal{E} \otimes \mathcal{D}$ denotes the projective tensor product ($\pi$-product) with another such space $\mathcal{E}$, $\mathcal{D}' \supseteq \mathcal{H}$ denotes the dual space of continuous antilinear functionals $\eta' : \eta \in \mathcal{D} \mapsto \langle \eta | \eta' \rangle$, with respect to the canonical pairing $\langle \eta | \eta' \rangle$ given by $\|\eta\|^2$ for $\eta' = \eta \in \mathcal{H}$, $\mathcal{B}(\mathcal{D})$ denotes the linear space of all continuous sesquilinear forms $\langle \eta | B\eta \rangle$ on $\mathcal{D}$, identified with the continuous linear operators $B : \mathcal{D} \to \mathcal{D}'$ (kernels), $B^{\dagger} \in \mathcal{B}(\mathcal{D})$ is the Hermit conjugated form (kernel) $\langle \eta | B^{\dagger}\eta \rangle = \langle \eta | B\eta \rangle^{*}$, and $\mathcal{L}(\mathcal{D}) \subseteq \mathcal{B}(\mathcal{D})$ denotes the algebra of all strongly continuous operators $B : \mathcal{D} \to \mathcal{D}$. Any such space $\mathcal{D}$ can be considered as a projective limit with respect to an increasing sequence of norms $\|\cdot\|_p > \|\cdot\|$ on $\mathcal{D}$; for the definitions and properties of this standard topological notions see for example [28]. The spaces $\mathcal{D}'$ and $\mathcal{B}(\mathcal{D})$ will be equipped with w*- topologies induced by their preduals $\mathcal{D}$ and $\mathcal{D} \otimes \mathcal{D}$, and coinciding with the weak topology on each bounded

subset with respect to a norm $\|\cdot\|_p$. Any operator $A \in \mathcal{L}(\mathcal{D})$ with $A^\dagger \in \mathcal{L}(\mathcal{D})$ can be uniquely extended to a weakly continuous operator onto $\mathcal{D}'$ as $A^{\dagger *}$, denoted again as $A$, where $A^*$ is the dual operator $\mathcal{D}' \to \mathcal{D}'$, $\langle \eta | A^* \eta' \rangle = \langle A\eta | \eta' \rangle$, defining the involution $A \mapsto A^*$ for the continuations $A : \mathcal{D}' \to \mathcal{D}'$. We say that the operator $A$ commutes with a sesquilinear form, $BA = AB$ if $\langle \eta | BA\eta \rangle = \langle A^\dagger \eta | B\eta \rangle$ for all $\eta \in \mathcal{D}$. The commutant $\mathcal{A}^c = \{B \in \mathcal{B}(\mathcal{D}) : [A, B] = 0, \forall A \in \mathcal{A}\}$ of an operator $*$-algebra $\mathcal{A} \subseteq \mathcal{L}(\mathcal{D})$ is weakly closed in $\mathcal{B}(\mathcal{D})$, so that the weak closure $\overline{\mathcal{B}} \subseteq \mathcal{B}(\mathcal{D})$ of any $\mathcal{B} \subseteq \mathcal{A}^c$ also commutes with $\mathcal{A}$.

Let us denote $\mathcal{B} = \mathcal{L}(\mathcal{H})$ the algebra of all bounded operators $B : \mathcal{H} \to \mathcal{H}$, $\|B\| < \infty$, $\overline{\mathcal{B}} = \mathcal{B}(\mathcal{D})$ means the weak closure of $\mathcal{B} \subseteq \mathcal{B}(\mathcal{D})$, and let $(\Omega, \mathfrak{A}, P)$ be a probability space with a filtration $(\mathfrak{A}_t)_{t>0}$, $\mathfrak{A}_t \subseteq \mathfrak{A}$ of $\sigma$-algebras on $\Omega$. One can assume that the filtration $\mathfrak{A}_t \subseteq \mathfrak{A}_s, \forall t < s$ is generated by the pieces $x_{t]} = \{r \mapsto x(r) : r \leq t\}$ of a stochastic process $x(t, \omega)$ with independent increments $\mathrm{d}x(t) = x(t + \Delta) - x(t)$, and the probability measure $P$ is invariant under the measurable representations $\omega \mapsto \omega_s \in \Omega$, $A_s^{-1} = \{\omega : \omega_s \in A\} \in \mathfrak{A}$, $\forall A \in \mathfrak{A}$ of the time shifts $t \mapsto t + s, s > 0$ on $\Omega \ni \omega$, corresponding to the shifts of the random increments

$$\mathrm{d}x(t, \omega_s) = \mathrm{d}x(t + s, \omega), \quad \forall \omega \in \Omega, t \in \mathbb{R}_+.$$

The *stochastic dynamics* over $\mathcal{B}$ with respect to the process $x(t)$ is described by a cocycle flow $\phi = (\phi_t)_{t>0}$ of linear completely positive [29] w*-continuous stochastic adapted maps $\phi_t(\omega) : \mathcal{B} \to \overline{\mathcal{B}}$, $\omega \in \Omega$ such that the stochastic process $y_t(\omega) = \langle \eta | \phi_t(\omega, B)\eta \rangle$ is causally measurable for each $\eta \in \mathcal{D}$, $B \in \mathcal{B}$ in the sense that $y_t^{-1}(B) \in \mathfrak{A}_t$, $\forall t > 0$ and any Borel $B \subseteq \mathbb{C}$. The maps $\phi_t$ can be extended on the $\mathfrak{A}$-measurable functions $Y : \omega \mapsto Y(\omega)$ with values $Y(\omega) \in \overline{\mathcal{B}}$ as the normal maps $\phi_t[Y](\omega) = \overline{\phi}_t(\omega, Y(\omega_t))$ for almost all $\omega \in \Omega$, where the linear maps $\overline{\phi}_t : \overline{\mathcal{B}} \to \overline{\mathcal{B}}$ are defined by the normal extensions of $\phi_t$ from the positive cone $\mathcal{B}_+$ onto $\overline{\mathcal{B}}_+$, so that the cocycle condition $\phi_r(\omega) \circ \phi_s(\omega_r) = \phi_{r+s}(\omega)$, $\forall r, s > 0$ reads as the semigroup condition $\phi_r[\phi_s[Y]] = \phi_{r+s}[Y]$ of the extended maps. As it was noted in the previous section, the maps $\phi_t(\omega)$ are not considered to be normalized to the identity, and can be even unbounded, but in the case of filtering dynamics they are supposed to be normalized, $\phi_t(\omega, I) = R_t(\omega)$, to an operator-valued martingale $R_t = \epsilon_t[R_s] \geq 0$ with $R_0(\omega) = I$, or to a positive submartingale, $R_t \geq \epsilon_t[R_s], \forall s > t$ in the subfiltering case, where $\epsilon_t$ is the conditional expectation over $\omega$ with respect to $\mathfrak{A}_t$.

Now we give an algebraic generalization and a Fock space representation of the filtering (or subfiltering) CP flows for a commutative Itô algebra $\mathfrak{a}$, which was suggested in [10] even in the noncommutative case.

The role of the classical process $x(t)$ will play the quantum stochastic process

$$X(t) = A \otimes I + I \otimes \Lambda(t, a), \quad A \in \mathcal{A}, a \in \mathfrak{a}$$

parametrized by an Abelian $*$-subalgebra $\mathcal{A} \subset \mathcal{L}(\mathcal{D})$ and a commutative Itô algebra $\mathfrak{a}$. Here $\Lambda(t, a)$ is the process with independent increment on a dense subspace $\mathfrak{F} \subset \Gamma(\mathfrak{E})$ of the Fock space $\Gamma(\mathfrak{E})$ over the space $\mathfrak{E} = L^2_{\mathcal{E}}(\mathbb{R}_+)$ of all square-norm integrable $\mathcal{E}$-valued functions on $\mathbb{R}_+$, where $\mathcal{E}$ is a pre-Hilbert space of the representation $a \in \mathfrak{a} \mapsto (a^\mu_\nu)^{\mu=-,\bullet}_{\nu=+,\bullet}$ for the Itô $\star$-algebra $\mathfrak{a}$. Every commutative Itô $*$-algebra is the sum $\mathfrak{a} = \mathbb{C}d + \mathfrak{a}_0 + \mathfrak{a}_1$ of the Wiener and Poisson algebras $\mathbb{C}d + \mathfrak{a}_0$, $\mathbb{C}d + \mathfrak{a}_1$, such that each $a \in \mathfrak{a}$ has the unique decomposition $a = l(a)d + b + c$,

where $d$ is the death of $\mathfrak{a}$, $b \in \mathfrak{a}_0$ is defined by the conditions

$$b_+^- = l(b) = 0, \qquad b_\bullet^\bullet = j(b) = 0,$$

and $c \in \mathfrak{a}_1$ is orthogonal to $b$: $bc = 0 = cb$, defined by the condition $c_\bullet^\bullet = 0 \Rightarrow c_+^\bullet = 0 = c_\bullet^-$. Thus the space $\mathcal{E}$ is decomposed into the orthogonal sum $\mathcal{E}_0 \oplus \mathcal{E}_1 \oplus \mathcal{E}_\perp$ with $\mathcal{E}_0$ generated by $k(\mathfrak{a}_0)$, $\mathcal{E}_1$ generated by $k(\mathfrak{a}_1)$, and $\mathcal{E}_\perp$ is the orthogonal complement, which is zero if $\mathcal{E}$ is the minimal space of the representation of $\mathfrak{a}$.

Assuming that $\mathcal{E}$ is a Fréchet space, given by an increasing sequence of Hilbertian norms $\|e^\bullet\|(\xi) > \|e^\bullet\|$, $\xi \in \mathbb{N}$, we define $\mathfrak{F}$ as the projective limit $\cap_\xi \Gamma(\mathfrak{E}, \xi)$ of the Fock spaces $\Gamma(\mathfrak{E}, \xi) \subseteq \Gamma(\mathfrak{E})$, generated by coherent vectors $f^\otimes$, with respect to the norms

(2.1)
$$\left\|f^\otimes\right\|^2(\xi) = \int_\Gamma \left\|f^\otimes(\tau)\right\|^2(\xi)\,\mathrm{d}\tau := \sum_{n=0}^\infty \frac{1}{n!}\left(\int_0^\infty \|f^\bullet(t)\|^2(\xi)\,\mathrm{d}t\right)^n = e^{\|f^\bullet\|^2(\xi)}.$$

Here $f^\otimes(\tau) = \bigotimes_{t\in\tau} f^\bullet(t)$ for each $f^\bullet \in \mathfrak{E}$ is represented by tensor-functions on the space $\Gamma$ of all finite subsets $\tau = \{t_1, ..., t_n\} \subseteq \mathbb{R}_+$. Moreover, we shall assume that the Itô algebra $\mathfrak{a}$ is realized as a $\star$-subalgebra of Hudson-Parthasarathy (HP) algebra $\mathcal{Q}(\mathcal{E})$ of all quadruples $\boldsymbol{a} = (a_\nu^\mu)_{\nu=+,\bullet}^{\mu=-,\bullet}$ with $a_\bullet^\bullet \in \mathcal{L}(\mathcal{E})$, strongly representing the $\star$-semigroup $1 + \mathfrak{a}$ on the Fréchet space $\mathcal{E}$ by projective contractions $\delta_\bullet^\bullet + a_\bullet^\bullet \in \mathcal{L}(\mathcal{E})$ in the sense that for each $\zeta \in \mathbb{N}$ there exists $\xi$ such that $\|e^\bullet + a_\bullet^\bullet e^\bullet\|(\zeta) \le \|e^\bullet\|(\xi)$ for all $e^\bullet \in \mathcal{E}$. The following theorem proves that these are natural assumptions (which are not restrictive in the simple Fock scale for a finite dimensional $\mathfrak{a}$.)

**Proposition 1.** *The exponential operators $W(t,a) =: \exp[\Lambda(t,a)]$ : defined as the solutions to the quantum Itô equation*

(2.2)
$$\mathrm{d}W_t(g) = W_t(g)\,\mathrm{d}\Lambda(t, g(t)), \quad W_0(g) = I, g(t) \in \mathfrak{a}$$

*with $g(t) = a$, are strongly continuous, $W(t,a) \in \mathcal{L}(\mathfrak{F})$, iff all $\widehat{a}_\bullet^\bullet = \delta_\bullet^\bullet + a_\bullet^\bullet$ are projective contructions on $\mathcal{E}$. They give an analytic representation*

(2.3)
$$W(t, a \star a) = W(t,a)^* W(t,a), \quad W(t,0) = I, \quad W(t,d) = e^t I$$

*of the unital $\star$-semigroup $1 + \mathfrak{a}$ for the Itô $\star$-algebra $\mathfrak{a}$ with respect to the $\star$-product $a \star a = a + a^\star a + a^\star$.*

*Proof.* The solutions $W(t,a)$ are uniquely defined on the coherent vectors as analytic functions

(2.4)
$$W(t,a) f^\otimes(\tau) = \otimes_{r\in\tau}^{r<t}\left(\widehat{a}_\bullet^\bullet f^\bullet(r) + a_+^\bullet\right)\exp\left[\int_0^t \left(a_\bullet^- f^\bullet(r) + a_+^-\right)\mathrm{d}r\right] \otimes_{r\in\tau}^{r\ge t} f^\bullet(r),$$

which obey the properties (2.3), see for example [3]. Thus the span of coherent vectors is invariant, and it is also invariant under $W(t,a)^* = W(t, a^\star)$. They can be extended on $\mathfrak{F}$ by continuity which follows from the continuity of Wick exponentials $\otimes\widehat{a}_\bullet^\bullet$ for the projective contractions $\widehat{a}_\bullet^\bullet \in \mathcal{E}$ , and boundedness of $a_+^\bullet \in \mathcal{E}$, $a_\bullet^- \in \mathcal{E}'$. ∎

Let $\mathfrak{D}$ denote the Fréchet space $\mathcal{D} \otimes \mathfrak{F}$, generated by $\psi = \eta \otimes f^\otimes$, $\eta \in \mathcal{D}$, $f^\bullet \in \mathfrak{E}$. Assuming for simplicity the separability of the Itô algebra in the sense $\mathcal{E} \subseteq \ell^2$ such that $f^\bullet = (f^m)^{m\in\mathbb{N}}$, one can identify each $\psi' \in \mathfrak{D}'$ with a sequence of $\mathcal{D}'$-valued symmetric tensor-functions $\psi'_{m_1,...,m_n}(t_1, ...t_n)$, $n = 0, 1, 2, ...$ . Let $(\mathfrak{D}_t)_{t>0}$ be the natural filtration and $\left(\mathfrak{D}_{[t}\right)_{t>0}$ be the backward filtration of the subspaces

$\mathfrak{D}_t = \mathcal{D} \otimes \mathfrak{F}_t$, $\mathfrak{D}_{[t} = \mathcal{D} \otimes \mathfrak{F}_{[t}$ generated by $\eta \otimes f^{\otimes}$ with $f^{\bullet} \in \mathfrak{E}_t$ and $f^{\bullet} \in \mathfrak{E}_{[t}$ respectively, where $\mathfrak{E}_t = L^2_{\mathcal{E}}[0,t)$, $\mathfrak{E}_{[t} = L^2_{\mathcal{E}}[t,\infty)$ are embedded into $\mathfrak{E}$. The spaces $\mathfrak{D}_t$, $\mathfrak{D}_{[t}$ of the restrictions $E_t\psi = \psi|\Gamma_t$, $E_{[t}\psi = \psi|\Gamma_{[t}$ onto $\Gamma_t = \{\tau_t = \tau \cap [0,t)\}$, $\Gamma_{[t} = \left\{\tau_{[t} = \tau \cap [t,\infty)\right\}$ are embedded into $\mathfrak{D}$ by the isometries $E_t^{\dagger} : \psi \mapsto \psi_t$, $E_{[t}^{\dagger} : \psi \mapsto \psi_{[t}$ as $\psi_t(\tau) = \psi(\tau_t)\,\delta_{\emptyset}\left(\tau_{[t}\right)$, $\psi_{[t}(\tau) = \delta_{\emptyset}(\tau_t)\,\psi\left(\tau_{[t}\right)$, where $\delta_{\emptyset}(\tau) = 1$ if $\tau = \emptyset$, otherwise $\delta_{\emptyset}(\tau) = 0$. The projectors $E_t, E_{[t}$ onto $\mathfrak{D}_t, \mathfrak{D}^t$ are extended onto $\mathfrak{D}'$ as the adjoints to $E_t^{\dagger}$, $E_{[t}^{\dagger}$. The time shift on $\mathfrak{D}'$ is defined by the semigroup $(T^t)_{t>0}$ of adjoint operators $T^t = T_t^*$ to $T_t\psi(\tau) = \psi(\tau + t)$, where $\tau{+}t = \{t_1 + t, ..., t_n + t\}$, $\emptyset{+}t = \emptyset$, such that $T^t\psi(\tau) = \delta_{\emptyset}(\tau_t)\,\psi\left(\tau_{[t} - t\right)$ are isometries for $\psi \in \mathfrak{D}$ onto $\mathfrak{D}_{[t}$. A family $(Z_t)_{t>0}$ of sesquilinear forms $\langle\psi|Z_t\psi\rangle$ given by linear operators $Z_t : \mathfrak{D} \to \mathfrak{D}'$ is called *adapted* (and $(Z^t)_{t>0}$ is called *backward adapted*) if

$$Z_t\left(\eta \otimes f^{\otimes}\right) = \psi' \otimes E_{[t}f^{\otimes} \quad \left(Z^t\left(\eta \otimes f^{\otimes}\right) = \psi' \otimes E_t f^{\otimes}\right), \quad \forall \eta \in \mathcal{D}, f^{\bullet} \in \mathfrak{E}, \tag{2.5}$$

where $\psi' \in \mathfrak{D}'_t$ $(\mathfrak{D}'_{[t})$ and $E_{[t}$ $(E_t)$ are the projectors onto $\mathfrak{F}_{[t}$ $(\mathfrak{F}_t)$ correspondingly.

The (vacuum) *conditional expectation* on $\mathcal{B}(\mathfrak{D})$ with respect to the past up to a time $t \in \mathbb{R}_+$ is defined as a positive projector, $\epsilon_t(Z) \geq 0$, if $Z \geq 0$, $\epsilon_t = \epsilon_t{\circ}\epsilon_s, \forall s > t$, giving an adapted sesquilinear form $Z_t = \epsilon_t(Z)$ in (2.5) for each $Z \in \mathcal{B}(\mathfrak{D})$ by $\psi' = E_t Z E_t^{\dagger}\psi$, where $\psi = \eta \otimes E_t f^{\otimes}$. The time shift $\left(\theta^t\right)_{t>0}$ on $\mathcal{B}(\mathfrak{D})$ is uniquely defined by the covariance condition $\theta^t(Z)\,T^t = T^t Z$ as a backward adapted family $Z^t = \theta^t(Z), t > 0$ for each $Z \in \mathcal{B}(\mathfrak{D})$. As in the bounded case [5] between the maps $\epsilon_t$ and $\theta^t$ we have the relation $\theta^r \circ \epsilon_s = \epsilon_{r+s} \circ \theta^r$ which follows from the operator relation $T^r E_s = E_{r+s}T^r$. An adapted family $(M_t)_{t>0}$ of positive $\langle\psi|M_t\psi\rangle \geq 0, \forall\psi \in \mathfrak{D}$ Hermitian $M_t^{\dagger} = M_t$ forms $M_t \in \mathcal{B}(\mathfrak{D})$ is called *martingale* (*submartingale*) if $\epsilon_t(M_s) = M_t$ $(\epsilon_t(M_s) \leq M_t)$ for all $s \geq t \geq 0$.

Let $\mathfrak{B}$ denote the space of all $Y \in \mathcal{B}(\mathfrak{D})$, commuting with all $X = \{X(t)\}$ in the sense

$$AY = YA, \quad \forall A \in \mathcal{A}, \quad YW(t,a) = W(t,a)\,Y, \quad \forall t > 0, a \in \mathfrak{a},$$

where $A(\eta \otimes \varphi) = A\eta{\otimes}\varphi$, $W(\eta \otimes \varphi) = \eta{\otimes}W\varphi$, and the unital $*$-algebra $\mathcal{B} \subseteq \mathcal{L}(\mathcal{H})$ be weakly dense in the commutant $\mathcal{A}^c$ (we can take $\mathcal{B} = \mathcal{L}(\mathcal{H})$ only if $\mathcal{A} = 0$, corresponding to $X(0) = 0$.) The quantum filtration $(\mathfrak{B}_t)_{t>0}$ is defined as the increasing family of subspaces $\mathfrak{B}_t \subseteq \mathfrak{B}_s, t \leq s$ of the adapted sesquilinear forms $Y_t \in \mathfrak{B}$. The covariant shifts $\theta^t : Y \mapsto Y^t$ leave the space $\mathfrak{B}$ invariant, mapping it onto the subspaces of backward adapted sesquilinear forms $Y^t = \theta^t(Y)$.

The *quantum stochastic positive flow* over $\mathcal{B}$ is described by a one parameter family $\phi = (\phi_t)_{t>0}$ of linear w*-continuous maps $\phi_t : \mathcal{B} \to \mathfrak{B}$ satisfying

(1) the causality condition $\phi_t(B) \subseteq \mathfrak{B}_t, \quad \forall B \in \mathcal{B}, t \in \mathbb{R}_+$,
(2) the complete positivity condition $[\phi_t(B_{kl})] \geq 0$ for each $t > 0$ and for any positive definite matrix $[B_{kl}] \geq 0$ with $B_{kl} \in \mathcal{B}$,
(3) the cocycle condition $\phi_r \circ \phi_s^r = \phi_{r+s}, \forall t, s > 0$ with respect to the covariant shift $\phi_s^r = \theta^r \circ \phi_s$.

Here the composition $\circ$ is understood as $\phi_r[\phi_s(B)] = \phi_{r+s}(B)$ in terms of the linear normal extensions of $\phi_t[B \otimes Z] = \overline{\phi}_t(B)\,Z^t$ to the CP maps $\mathfrak{B} \to \mathfrak{B}$, forming a one-parameter semigroup, where $B \in \overline{\mathcal{B}}$, $\overline{\phi}_t : \overline{\mathcal{B}} \to \mathfrak{B}_t$ are the normal extensions of $\phi_t$, $Z^t = \theta^t(Z)$, $Z \in \mathcal{B}(\mathfrak{F})$. These can be defined like in classical case as $\phi_t[Y]\left(\bar{f}^{\bullet}, f^{\bullet}\right) = \overline{\phi}_t\left(\bar{f}^{\bullet}, Y\left(\bar{f}_t^{\bullet}, f_t^{\bullet}\right), f^{\bullet}\right)$ with $f_t^{\bullet}(r) = f^{\bullet}(t + r)$ by the

coherent matrix elements $Y\left(\bar{f}^{\bullet}, f^{\bullet}\right) = F^{*}YF$ for $Y \in \mathfrak{B}$ given by the continuous operators $F : \eta \mapsto \psi_{f} = \eta \otimes f^{\otimes}$ , $\eta \in \mathcal{D}$ for each $f^{\bullet} \in \mathfrak{E}_{t}$ with the adjoints $F^{*}\psi' = \int_{\tau<t} f^{\otimes}(\tau)^{*} \psi'(\tau)\,\mathrm{d}\tau$ for $\psi' \in \mathfrak{D}'$.

The flow is called *(sub)-filtering*, if $R_{t} = \phi_{t}(I)$ is a (sub)-martingale with $R_{0} = I$, and is called contractive, if $I \geq R_{t} \geq R_{s}$ for all $0 \leq t \leq s \in \mathbb{R}_{+}$.

**Proposition 2.** *The complete positivity for adapted linear maps $\phi_{t} : \mathcal{B} \to \mathcal{B}(\mathfrak{D})$ can be written as*

(2.6)
$$\sum_{f,h\in\mathfrak{E}_{t}} \sum_{B,C\in\mathcal{B}} \langle \xi_{B}^{f} | \phi_{t}\left(\bar{f}^{\bullet}, B^{*}C, h^{\bullet}\right) \xi_{C}^{h} \rangle := \langle \eta^{k} | \phi_{t}\left(\bar{f}_{k}^{\bullet}, B_{k}^{*}B_{l}, h_{l}^{\bullet}\right) \eta^{l} \rangle \geq 0, \quad \forall t > 0$$

*(the usual summation rule over repeated cross-level indices is understood), where $\xi_{B}^{f} = \eta^{k}$ if $f^{\bullet} = f_{k}^{\bullet}$ and $B = B_{k}$ with $f_{k}^{\bullet} \in \mathfrak{E}_{t}, B_{k} \in \mathcal{B}$, $k = 1, 2, ...$, otherwise $\xi_{B}^{f} = 0$, and $\phi_{t}(B, f^{\bullet}) = \phi_{t}(B)F$, $\phi_{t}\left(\bar{f}^{\bullet}, B\right) = F^{*}\phi_{t}(B)$.*

*Proof.* By definition the map $\phi$ into the sesquilinear forms is completely positive on $\mathcal{B}$ if $\left\langle \psi^{k} | \phi(B_{kl}) \psi^{l} \right\rangle \geq 0$ whenever $\langle \eta^{k} | B_{kl} \eta^{l} \rangle \geq 0$, where $\eta^{k}, \psi^{k}$ are arbitrary finite sequences. Approximating from below the latter positive forms by sums of the forms $\sum_{kl} \langle \eta^{k} | B_{ik}^{*} B_{il} \eta^{l} \rangle \geq 0$, the complete positivity can be tested only for the forms $\sum_{kl} \langle \eta^{k} | B_{k}^{*} B_{l} \eta^{l} \rangle \geq 0$ due to the additivity $\phi\left(\sum_{i} B_{ik}^{*} B_{il}\right) = \sum_{i} \phi\left(B_{ik}^{*} B_{il}\right)$. If $\phi_{t}$ is adapted, this can be written as

$$\sum_{B,C\in\mathcal{B}} \langle \chi_{B} | \phi(B^{*}C) \chi_{C} \rangle = \left\langle \psi^{k} | \phi(B_{k}^{*}B_{l}) \psi^{l} \right\rangle := \sum_{k,l} \left\langle \psi^{k} | \phi(B_{k}^{*}B_{l}) \psi^{l} \right\rangle \geq 0,$$

where $\chi_{B} = \psi^{k} \in \mathfrak{D}_{t}$ if $B = B_{k} \in \mathcal{B}$, otherwise $\chi_{B} = 0$. Because any $\psi \in \mathfrak{D}_{t}$ can be approximated by a $\mathcal{D}$-span $\sum_{f} \eta^{f} \otimes f^{\otimes}$of coherent vectors over $f_{k}^{\bullet} \in \mathfrak{E}_{t}$, it is sufficient to define the CP property only for such spans as

$$0 \leq \sum_{f,h} \sum_{B,C} \left\langle \xi_{B}^{f} \otimes f^{\otimes} | \phi(B^{*}C) \left( \xi_{C}^{h} \otimes h^{\otimes} \right) \right\rangle = \sum_{f,h} \sum_{B,C} \left\langle \xi_{B}^{f} | \phi\left(\bar{f}^{\bullet}, B^{*}C, h^{\bullet}\right) \xi_{C}^{h} \right\rangle.$$

∎

Note that the subfiltering (filtering) flows can be considered as a quantum stochastic CP dilations of the quantum sub-Markov (Markov) semigroups $\theta = (\theta_{t})_{t>0}$, $\theta_{r} \circ \theta_{s} = \theta_{r+s}$ in the sense $\theta_{t} = \epsilon \circ \phi_{t}$, where $\epsilon(Y)\eta = EY\psi_{0}$, $E\psi' = \psi'(\emptyset), \forall \psi' \in \mathfrak{D}'$, with $\theta_{s}(I) \leq \theta_{t}(I) \leq I$ $(\theta_{t}(I) = I)$, $\forall t \leq s$. The contraction $C_{t} = \theta_{t}(I)$ with $C_{0} = I$ defines the probability $\langle \eta | C_{t} \eta \rangle \leq 1$, $\forall \eta \in \mathcal{H}, \|\eta\| = 1$ for an unstable system not to be demolished by a time $t \in \mathbb{R}_{+}$, and the conditional expectations $\langle \eta | AC_{t}\eta \rangle / \langle \eta | C_{t}\eta \rangle$ of the initial nondemolition observables $A \in \mathcal{A}$ in any state $\eta \in \mathcal{D}$, and thus in any initial state $\psi_{0} \in \eta \otimes \delta_{\emptyset}$. The following theorem shows that the submartingale (or the contraction) $R_{t} = \phi_{t}(I)$ is the density operator with respect to $\psi_{0} = \eta \otimes \delta_{\emptyset}$, $\eta \in \mathcal{H}$ (or with respect to any $\psi \in \mathcal{H} \otimes \mathfrak{F}$) also for the conditional state of the restricted nondemolition process $X_{t]} = \{r \mapsto X(r) : r \leq t\}$.

**Theorem 3.** *Let $t \mapsto R_{t} \in \mathfrak{B}_{t}$ be a positive (sub)-martingale and $(\mathfrak{g}_{t})_{t>0}$ be the increasing family of $\star$-semigroups $\mathfrak{g}_{t}$ of step functions $g : \mathbb{R}_{+} \to \mathfrak{a}$, $g(s) = 0$, $\forall s \geq t$ under the $\star$-product*

$$(g_{k} \star g_{l})(t) = g_{l}(t) + g_{k}(t)^{\star} g_{l}(t) + g_{k}(t)^{\star} \tag{2.7}$$

*of $g_k^\star = g_k \star 0$ and $g_l = 0 \star g_l$. The generating function $\vartheta_t(g) = \epsilon[R_t W_t(g)]$ of the output state for the process $\Lambda(t)$, defined for any $g \in \mathfrak{g}_t$ and each $t > 0$ as*

$$\langle \eta | \vartheta_t(g)\,\eta\rangle = \langle \psi_0 | R_t W_t(g)\,\psi_0\rangle, \quad \psi_0 = \eta \otimes \delta_\emptyset, \tag{2.8}$$

*is $\mathcal{B}^c$-valued, positive, $\vartheta_t \geq 0$ in the sense of positive definiteness of the kernel*

$$\left\langle \eta^k | \vartheta_t(g_k \star g_l)\,\eta^l \right\rangle \geq 0, \quad \forall g_k \in \mathfrak{g}_t; \eta^k \in \mathcal{D}, \tag{2.9}$$

*and $\vartheta_t \geq \vartheta_s|\mathfrak{g}_t$ in this sense for any $s \geq t$. If $R_0 = I$, then $\vartheta_0(0) = I \geq \vartheta_t(0)$, and if $R_t$ is a martingale, then $\vartheta_t = \vartheta_s|\mathfrak{g}_t$ for any $s \geq t$, and $\vartheta_t(0) = I$ for all $t \in \mathbb{R}_+$. Any family $\vartheta = (\vartheta_t)_{t\geq 0}$ of positive-definite functions $\vartheta_t : \mathfrak{g}_t \to \mathcal{B}^c$, satisfying the above consistency and normalization properties, is the state generating function of the form (2.8) iff it is absolutely continuous in the following sense*

$$\lim_{n\to\infty} \sum_{g \in \mathfrak{g}_t} \eta_n^g \otimes g_+^\otimes = 0 \Rightarrow \lim_{n\to\infty} \sum_{g,h \in \mathfrak{g}_t} \left\langle \eta_n^g | \vartheta_t(g \star h)\,\eta_n^h \right\rangle = 0, \tag{2.10}$$

*where $g_+^\otimes(\tau) = \otimes_{t\in\tau} g_+^\bullet(t)$ and $\eta_n^g = 0$ for almost all $g$ (i.e. except for a finite number of $g \in \mathfrak{g}_t$).*

The proof is given in [10] even for the general (noncommutative) algebras $\mathcal{A}$ and $\mathfrak{a}$.

## 3. Generators of quantum CP dynamics

The quantum stochastically differentiable positive flow $\phi$ is defined as a weakly continuous function $t \mapsto \phi_t$ with CP values $\phi_t : \mathcal{B} \to \mathfrak{B}_t$, $\phi_0(B) = B \otimes I, \forall B \in \mathcal{B}$ such that for any product-vector $\psi_f = \eta \otimes f^\otimes$ given by $\eta \in \mathcal{D}$ and $f^\bullet \in \mathfrak{E}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \psi_f | \phi_t(B)\,\psi_f \right\rangle = \left\langle \psi_f | \phi_t\left(\lambda\left(\bar{f}^\bullet(t), B, f^\bullet(t)\right)\right)\psi_f \right\rangle, \qquad B \in \mathcal{B}, \tag{3.1}$$

where $\lambda(\bar{e}^\bullet, B, e^\bullet) = \lambda(B) + e_\bullet \lambda^\bullet(B) + \lambda_\bullet(B)\,e^\bullet + e_\bullet \lambda_\bullet^\bullet(B)\,e^\bullet$, $e_\bullet = \bar{e}^\bullet$ is the linear form on $\mathcal{E}$ with $e_\bullet^* = e^\bullet \in \mathcal{E}$ and $\left\langle \psi_f | \phi_0(B)\,\psi_f \right\rangle = \langle \eta | B\eta \rangle \exp \|f^\bullet\|^2$. The generator $\lambda(B) = \lambda(0, B, 0)$ of the quantum dynamical semigroup $\theta_t = \epsilon \circ \phi_t$ is a linear w*-continuous map $B \mapsto \lambda(B) \in \mathcal{A}^c$, $\lambda^\bullet = \lambda_\bullet^\dagger$ is a linear w*-continuous map given by the Hermitian adjoint values $\lambda_\bullet(B^*) = \lambda^\bullet(B)^\dagger$ in the continuous operators $\mathcal{E} \to \mathcal{A}^c$, and $\lambda_\bullet^\bullet : \mathcal{B} \to \mathcal{B}(\mathcal{D} \otimes \mathcal{E})$is a w*-continuous map with the values $\lambda_\bullet^\bullet(B)$ given by continuous operators $\mathcal{E} \otimes \mathcal{E} \to \mathcal{A}^c$. The differential evolution equation (3.1) for the coherent vector matrix elements $\left\langle \psi_f | \phi_t(B)\,\psi_f \right\rangle$ corresponds to the Itô form [6] of the quantum stochastic equation

$$\mathrm{d}\phi_t(B) = \phi_t \circ \lambda_\nu^\mu(B)\,\mathrm{d}\Lambda_\mu^\nu := \sum_{\mu,\nu} \phi_t(\lambda_\nu^\mu(B))\,\mathrm{d}\Lambda_\mu^\nu, \qquad B \in \mathcal{B} \tag{3.2}$$

with the initial condition $\phi_0(B) = B$, for all $B \in \mathcal{B}$. Here $\lambda_\nu^\mu$ are the flow generators $\lambda_+^- = \lambda$, $\lambda_+^\bullet = \lambda^\bullet$, $\lambda_\bullet^- = \lambda_\bullet$, $\lambda_\bullet^\bullet$, called the structural maps, and the summation is taken over the indices $\mu = -, \bullet$, $\nu = +, \bullet$ of the standard quantum stochastic integrators $\Lambda_\mu^\nu$. For simplicity we shall assume that the pre-Hilbert Fréchet space $\mathcal{E}$ is separable, $\mathcal{E} \subseteq \ell^2$. Then the index $\bullet$ can take any value in $\{1, 2, ...\}$ and $\Lambda_\mu^\nu(t)$ are indexed with $\mu \in \{-, 1, 2, ...\}$, $\nu \in \{+, 1, 2, ...\}$ as the standard time $\Lambda_-^+(t) = t\mathrm{I}$, annihilation $\Lambda_-^m(t)$, creation $\Lambda_n^+(t)$ and exchange-number $\Lambda_n^m(t)$ operator integrators with $m, n \in \mathbb{N}$. The infinitesimal increments $\mathrm{d}\Lambda_\nu^\mu(t) = \Lambda_\nu^{t\mu}(\mathrm{d}t)$ are formally

defined by the HP multiplication table [6] and the $\star$ -property [15],

$$\mathrm{d}\Lambda_{\mu}^{\alpha}\mathrm{d}\Lambda_{\beta}^{\nu}=\delta_{\beta}^{\alpha}\mathrm{d}\Lambda_{\mu}^{\nu},\qquad\Lambda^{\star}=\Lambda, \tag{3.3}$$

where $\delta_{\beta}^{\alpha}$ is the usual Kronecker delta restricted to the indices $\alpha\in\{-,1,2,...\}$, $\beta\in\{+,1,2,...\}$ and $\Lambda_{-\nu}^{\star\mu}=\Lambda_{-\mu}^{\nu*}$ with respect to the reflection $-(-)=+,-(+)=-$ of the indices $(-,+)$ only.

The linear equation (3.2) of a particular type, (quantum Langevin equation) with bounded finite-dimensional structural maps $\lambda_{\nu}^{\mu}$ was introduced by Evans and Hudson [2] in order to describe the $*$-homomorphic quantum stochastic evolutions. The constructed quantum stochastic $*$-homomorphic flow (EH-flow) is identity preserving and is obviously completely positive, but it is hard to prove these algebraic properties for the unbounded case. However the typical quantum filtering dynamics is not homomorphic or identity preserving, but it is completely positive and in the most interesting cases is described by unbounded generators $\lambda_{\nu}^{\mu}$. In the general content the equation (3.2) was studied in [31], and the correspondent quantum stochastic, not necessarily homomorphic and normalized flow was constructed even for the infinitely-dimensional non-adapted case under the natural integrability condition for the chronological products of the generators $\lambda_{\nu}^{\mu}$ in the norm scale (2.1). The EH flows with unbounded $\lambda_{\nu}^{\mu}$, satisfying certain analyticity conditions, have been recently constructed in strong sense by Fagnola-Sinha in [30] for the non-Hilbert class $L^{\infty}$ of test functions $f^{\bullet}$. Here we will formulate the necessary differential conditions which follow from the complete positivity, causality, and martingale properties of the filtering flows, and which are sufficient for the construction of the quantum stochastic flows obeying these properties in the case of the bounded $\lambda_{\nu}^{\mu}$. As we showed in [7], the found properties are sufficient to define the general structure of the bounded generators, and this structure will help us in construction of the minimal completely positive weak solutions for the quantum filtering equations also with unbounded $\lambda_{\nu}^{\mu}$.

Obviously the linear w*-continuous generators $\lambda_{\nu}^{\mu}:\mathcal{B}\rightarrow\mathcal{A}^{c}$ for CP flows $\phi_{t}^{*}=\phi_{t}$, where $\phi_{t}^{*}(B)=\phi_{t}(B^{*})^{\dagger}$, must satisfy the $\star$ -property $\lambda^{\star}=\lambda$, where $\lambda_{-\mu}^{\star\nu}=\lambda_{-\nu}^{\mu*}$, $\lambda_{\nu}^{\mu*}(B)=\lambda_{\nu}^{\mu}(B^{*})^{*}$ and are independent of $t$, corresponding to cocycle property $\phi_{s}\circ\phi_{r}^{s}=\phi_{s+r}$, where $\phi_{t}^{s}$ is the solution to (3.2) with $\Lambda_{\nu}^{\mu}(t)$ replaced by $\Lambda_{\nu}^{s\mu}(t)$, and $\lambda_{+}^{-}(I)=0$ if $\phi$ is a filtering flow, $\phi_{t}(I)=I$, as it is in the multiplicative case [2]. We shall assume that $\boldsymbol{\lambda}=(\lambda_{\nu}^{\mu})_{\nu=+,\bullet}^{\mu=-,\bullet}$ for each $B^{*}=B$ defines a continuous Hermitian form $\boldsymbol{b}=\boldsymbol{\lambda}(B)$ on the Fréchet space $\mathcal{D}\oplus\mathcal{D}_{\bullet}$,

$$\langle\boldsymbol{\eta}|\,\boldsymbol{b}\,\boldsymbol{\eta}\rangle=\sum_{m,n}\langle\eta^{m}|b_{n}^{m}\eta^{n}\rangle+\sum_{m}\left\langle\eta^{m}|b_{+}^{m}\eta\right\rangle+\sum_{n}\left\langle\eta|b_{n}^{-}\eta^{n}\right\rangle+\left\langle\eta|b_{+}^{-}\eta\right\rangle,$$

where $\eta\in\mathcal{D}$, $\eta^{\bullet}=(\eta^{m})^{m\in\mathbb{N}}\in\mathcal{D}_{\bullet}=\mathcal{D}\otimes\mathcal{E}$. We say that an Itô algebra $\mathfrak{a}$ , represented on $\mathcal{E}$, commutes in HP sense with a $\boldsymbol{b}$, given by the form-generator $\boldsymbol{\lambda}$ if $(I\otimes a_{\bullet}^{\mu})\,b_{\nu}^{\bullet}=b_{\bullet}^{\mu}\,(I\otimes a_{\nu}^{\bullet})$ (For simplicity the ampliation $I\otimes a_{\nu}^{\mu}$ will be written again as $a_{\nu}^{\mu}$.) Note that if we define the matrix elements $a_{\nu}^{\mu}$, $b_{\nu}^{\mu}$ also for $\mu=+$ and $\nu=-$, by the extension

$$a_{\nu}^{+}=0=a_{-}^{\mu},\qquad\lambda_{\nu}^{+}(B)=0=\lambda_{-}^{\mu}(B)\,,\quad\forall a\in\mathfrak{a},B\in\mathcal{B},$$

the HP product (0.3) of $\boldsymbol{a}$ and $\boldsymbol{b}$ can be written in terms of the usual matrix product $\mathbf{ab}=\left[a_{\lambda}^{\mu}b_{\nu}^{\lambda}\right]$ of the extended quadratic matrices $\mathbf{a}=[a_{\nu}^{\mu}]_{\nu=-,\bullet,+}^{\mu=-,\bullet,+}$ and $\mathbf{b}=\boldsymbol{b}\mathbf{g}$, where $\mathbf{g}=\left[\delta_{-\nu}^{\mu}\right]$. Then one can extend the summation in (3.2) so it is

also over $\mu = +$, and $\nu = -$, such that $b^{\mu}_{\nu}\mathrm{d}\Lambda^{\nu}_{\mu}$ is written as the trace $\mathbf{b}\cdot\mathrm{d}\boldsymbol{\Lambda}$ over all $\mu, \nu$. By such an extension the multiplication table for $\mathrm{d}\Lambda(a) = \mathbf{a}\cdot\mathrm{d}\boldsymbol{\Lambda}$ , $\mathrm{d}\Lambda(b) = \mathbf{b}\cdot\mathrm{d}\boldsymbol{\Lambda}$ can be represented as $\mathrm{d}\Lambda(a)\,\mathrm{d}\Lambda(b) = \mathbf{ab}\cdot\mathrm{d}\boldsymbol{\Lambda}$, and the involution $\mathbf{b} \mapsto \mathbf{b}^{\star}$, defining $\mathrm{d}\Lambda(b)^{\dagger} = \mathbf{b}^{\star}\cdot\mathrm{d}\boldsymbol{\Lambda}$, can be obtained by the pseudo-Hermitian conjugation $b^{\star\nu}_{\alpha} = g_{\alpha\mu}b^{\mu *}_{\beta}g^{\beta\nu}$ respectively to the indefinite Minkowski metric tensor $\mathbf{g} = [g_{\mu\nu}]$ and its inverse $\mathbf{g}^{-1} = [g^{\mu\nu}]$, given by $g^{\mu\nu} = \delta^{\mu}_{-\nu}I = g_{\mu\nu}$.

Now let us find the differential form of the normalization and causality conditions with respect to the quantum stationary process, with independent increments $\mathrm{d}X(t) = X(t+\Delta) - X(s)$ generated by an Itô algebra $\mathfrak{a}$ on the separable space $\mathcal{E}$.

**Proposition 4.** *Let $\phi$ be a flow, satisfying the quantum stochastic equation (3.2), and $[W_t(g), \phi_t(B)] = 0$ for all $g \in \mathfrak{g}, B \in \mathcal{B}$. Then the coefficients $b^{\mu}_{\nu} = \lambda^{\mu}_{\nu}(B)$, $\mu = -, \bullet$, $\nu = +, \bullet$, where $\bullet = 1, 2, \ldots$, written in the matrix form $\boldsymbol{b} = (b^{\mu}_{\nu})^{\mu=-,\bullet}_{\nu=+,\bullet}$, commute in the sense of the HP product with $\boldsymbol{a} = (a^{\mu}_{\nu})^{\mu=-,\bullet}_{\nu=+,\bullet}$ for all $a \in \mathfrak{a}$ and $B \in \mathcal{B}$:*

$$[\boldsymbol{a}, \boldsymbol{b}] := (a^{\mu}_{\bullet}b^{\bullet}_{\nu} - b^{\mu}_{\bullet}a^{\bullet}_{\nu})^{\mu=-,\bullet}_{\nu=+,\bullet} = 0. \tag{3.4}$$

*Proof.* Since $\epsilon_t(\phi_s(I) - \phi_t(I))$ is a negative Hermitian form,

$$\epsilon_t(\mathrm{d}\phi_t(I)) = \epsilon_t\left(\phi_t(\lambda^{\mu}_{\nu}(I))\,\mathrm{d}\Lambda^{\nu}_{\mu}\right) = \phi_t\left(\lambda^{-}_{+}(I)\right)\mathrm{d}t \leq 0.$$

Since $Y_t = \phi_t(B)$ commutes with $W_t(g)$ for all $B$ and $g(t) = a$, we have by virtue of quantum Itô's formula

$$\mathrm{d}[Y_t, W_t] = [\mathrm{d}Y_t, W_t] + [Y_t, \mathrm{d}W_t] + [\mathrm{d}Y_t, \mathrm{d}W_t] = 0.$$

The equations (2.2), (3.2) and commutativity of $a^{\mu}_{\nu}$ with $Y_t$ and $W_t$ imply

$$\begin{aligned} & ([\phi_t(b^{\mu}_{\nu}), W_t] + [Y_t, a^{\mu}_{\nu}W_t] + \phi_t(b^{\mu}_{\bullet})\,a^{\bullet}_{\nu}W_t - a^{\mu}_{\bullet}W_t\phi_t(b^{\bullet}_{\nu}))\,\mathrm{d}\Lambda^{\nu}_{\mu} \\ = \quad & W_t(\phi_t(b^{\mu}_{\bullet})\,a^{\bullet}_{\nu} - a^{\mu}_{\bullet}\phi_t(b^{\bullet}_{\nu}))\,\mathrm{d}\Lambda^{\mu}_{\nu} = W_t\phi_t(b^{\mu}_{\bullet}a^{\bullet}_{\nu} - a^{\mu}_{\bullet}b^{\bullet}_{\nu})\,\mathrm{d}\Lambda^{\nu}_{\mu.} = 0. \end{aligned}$$

Thus $\boldsymbol{a} \bullet \boldsymbol{b} = \boldsymbol{b} \bullet \boldsymbol{a}$ by the argument [4] of independence of the integrators $\mathrm{d}\Lambda^{\nu}_{\mu}$. ∎

In order to formulate the CP differential condition we need the notion of *quantum stochastic germ* for the CP flow $\phi$ at $t = 0$. It was defined in [31, 9], for a quantum stochastic differential (3.2) with $\phi_0(B) = B, \forall B \in \mathcal{B}$ as $\gamma^{\mu}_{\nu} = \lambda^{\mu}_{\nu} + \imath^{\mu}_{\nu}$, where $\lambda^{\mu}_{\nu}$ are the structural maps $B \mapsto \lambda^{\mu}_{\nu}(B)$ given by the generators of the quantum Itô equation (3.2) and $\imath^{\mu}_{\nu} : B \mapsto B\delta^{\mu}_{\nu}$ is the ampliation of $\mathcal{B}$. Let us prove that the germ-maps $\gamma^{\mu}_{\nu}$ of a CP flow $\phi$ must be conditionally completely positive (CCP) in a degenerated sense as it was found for the finite-dimensional bounded case in [7, 11]. Another, equivalent, but not so explicit characterization was suggested for this particular case in [12].

**Theorem 5.** *If $\phi$ is a completely positive flow satisfying the quantum stochastic equation (3.2) with $\phi_0(B) = B$, then the germ-matrix $\boldsymbol{\gamma} = (\lambda^{\mu}_{\nu} + \imath^{\mu}_{\nu})^{\mu=-,\bullet}_{\nu=+,\bullet}$ is conditionally completely positive in the sense*

$$\sum_{B \in \mathcal{B}} \boldsymbol{\iota}(B)\,\boldsymbol{\zeta}_B = 0 \Rightarrow \sum_{B,C \in \mathcal{B}} \langle \boldsymbol{\zeta}_B | \boldsymbol{\gamma}(B^{*}C)\,\boldsymbol{\zeta}_C \rangle \geq 0.$$

*Here $\boldsymbol{\zeta} \in \mathcal{D} \oplus \mathcal{D}_{\bullet}, \mathcal{D}_{\bullet} = \mathcal{D} \otimes \mathcal{E}$, and $\boldsymbol{\iota} = (\imath^{\mu}_{\nu})^{\mu=-,\bullet}_{\nu=+,\bullet}$ is the degenerate representation $\imath^{\mu}_{\nu}(B) = B\delta^{+}_{\nu}\delta^{\mu}_{-}$, written both with $\boldsymbol{\gamma}$ in the matrix form as*

$$\boldsymbol{\gamma} = \begin{pmatrix} \gamma & \gamma_{\bullet} \\ \gamma^{\bullet} & \gamma^{\bullet}_{\bullet} \end{pmatrix}, \qquad \boldsymbol{\iota}(B) = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}, \tag{3.5}$$

*where* $\gamma=\lambda_{+}^{-}$, $\gamma^{m}=\lambda_{+}^{m}$, $\gamma_{n}=\lambda_{n}^{-}$, $\gamma_{n}^{m}=\imath_{n}^{m}+\lambda_{n}^{m}$ *with* $\imath_{n}^{m}(B)=B\delta_{n}^{m}$ *such that*

$$\gamma\left(B^{*}\right)=\gamma\left(B\right)^{*},\qquad\gamma^{n}\left(B^{*}\right)=\gamma_{n}\left(B\right)^{*},\qquad\gamma_{n}^{m}\left(B^{*}\right)=\gamma_{m}^{n}\left(B\right)^{*}.\tag{3.6}$$

*If* $\phi$ *is subfiltering, then* $D=-\lambda_{+}^{-}(I)$ *is a positive Hermitian form,* $\langle\eta|D\eta\rangle\geq0$, *for all* $\eta\in\mathcal{D}$, *and if* $\phi$ *is contractive, then* $\boldsymbol{D}=-\boldsymbol{\lambda}(I)$ *is positive in the sense* $\langle\boldsymbol{\eta}|\boldsymbol{D}\boldsymbol{\eta}\rangle\geq0$ *for all* $\boldsymbol{\eta}\in\mathcal{D}\oplus\mathcal{D}_{\bullet}$.

The proof is given in [9, 10] even for the general (noncommutative) algebras $\mathfrak{a}$ and $\mathcal{A}$.

Obviously the CCP property for the germ-matrix $\boldsymbol{\gamma}$ is invariant under the transformation $\boldsymbol{\gamma}\mapsto\boldsymbol{\varphi}$ given by

$$\boldsymbol{\varphi}\left(B\right)=\boldsymbol{\gamma}\left(B\right)+\boldsymbol{\iota}\left(B\right)\boldsymbol{K}+\boldsymbol{K}^{*}\boldsymbol{\iota}\left(B\right),\tag{3.7}$$

where $\boldsymbol{K}=(K_{\nu}^{\mu})_{\nu=+,\bullet}^{\mu=-,\bullet}$ is an arbitrary matrix of $K_{\nu}^{\mu}\in\mathcal{L}(\mathcal{D})$ with $K_{-\nu}^{*\mu}=K_{-\mu}^{\nu*}$. As was proven in [7, 11] for the case of finite-dimensional matrix $\boldsymbol{\gamma}$ of bounded $\gamma_{\nu}^{\mu}$, see also [12], the matrix elements $K_{\nu}^{-}$ can be chosen in such way that the matrix map $\boldsymbol{\varphi}=(\varphi_{\nu}^{\mu})_{\nu=+,\bullet}^{\mu=-,\bullet}$ becomes CP from $\mathcal{B}$ into the quadratic matrices of $\varphi_{\nu}^{\mu}(B)$. (The other elements can be chosen arbitrarily, say as $K_{+}^{\bullet}=0$, $K_{\bullet}^{\bullet}=\frac{1}{2}I_{\bullet}^{\bullet}$, because (3.7) does not depend on $K_{+}^{\bullet},K_{\bullet}^{\bullet}$.) Thus the generator $\boldsymbol{\lambda}=\boldsymbol{\gamma}-\boldsymbol{\imath}$ for a quantum stochastic CP flow $\phi$ can be written (at least in the bounded case) as $\boldsymbol{\varphi}-\imath\boldsymbol{K}-\boldsymbol{K}^{*}\imath$:

$$\lambda_{\nu}^{\mu}\left(B\right)=\varphi_{\nu}^{\mu}\left(B\right)-B\left(\tfrac{1}{2}\delta_{\nu}^{\mu}I+\delta_{-}^{\mu}K_{\nu}\right)-\left(\tfrac{1}{2}\delta_{\nu}^{\mu}I+K^{\mu}\delta_{\nu}^{+}\right)B,\tag{3.8}$$

where $\varphi_{\nu}^{\mu}:\mathcal{B}\rightarrow\mathcal{B}(\mathcal{D})$ are matrix elements of the CP map $\boldsymbol{\varphi}$ and $K_{\nu}\in\mathcal{L}(\mathcal{D})$, $K^{-}=K_{+}^{*}$, $K^{m}=K_{m}^{*}$. Now we show that the germ-matrix of this form obeys the CCP property even in the general case of unbounded $K_{\nu}^{-}$, $\varphi_{\nu}^{\mu}(B)\in\mathcal{B}(\mathcal{D})$.

**Proposition 6.** *The matrix map* $\boldsymbol{\gamma}=(\gamma_{\nu}^{\mu})_{\nu=+,\bullet}^{\mu=-,\bullet}$ *given in (3.7) by*

$$\boldsymbol{\varphi}=\begin{pmatrix}\varphi&\varphi_{\bullet}\\\varphi^{\bullet}&\varphi_{\bullet}^{\bullet}\end{pmatrix},\quad\text{and}\quad\boldsymbol{K}=\begin{pmatrix}K&K_{\bullet}\\0&\frac{1}{2}I_{\bullet}^{\bullet}\end{pmatrix},\ \boldsymbol{K}^{*}=\begin{pmatrix}K^{*}&0\\K_{\bullet}^{*}&\frac{1}{2}I_{\bullet}^{\bullet}\end{pmatrix},\tag{3.9}$$

*with* $\varphi=\varphi_{+}^{-}$, $\varphi^{m}=\varphi_{+}^{m}$, $\varphi_{n}=\varphi_{n}^{-}$ *and* $\varphi_{n}^{m}=\gamma_{n}^{m}$ *is CCP with respect to the degenerate representation* $\boldsymbol{\iota}=\left(\delta_{-}^{\mu}\delta_{\nu}^{+}\iota\right)_{\nu=+,\bullet}^{\mu=-,\bullet}$, *where* $\iota(B)=B$, *if* $\boldsymbol{\varphi}$ *is a CP map.*

*Proof.* If $\boldsymbol{\iota}(B_{k})\boldsymbol{\eta}^{k}=0$, then

$$\begin{aligned}&\langle\boldsymbol{\eta}^{k}|\boldsymbol{\iota}\left(B_{k}^{*}B_{l}\right)\boldsymbol{K}+\boldsymbol{K}^{*}\boldsymbol{\iota}\left(B_{k}^{*}B_{l}\right)\boldsymbol{\eta}^{l}\rangle\\&=2\operatorname{Re}\left\langle\boldsymbol{\iota}\left(B_{k}\right)\boldsymbol{\eta}^{k}|\boldsymbol{\iota}\left(B_{l}\right)\boldsymbol{K}\boldsymbol{\eta}^{l}\right\rangle=0.\end{aligned}$$

Hence the CCP for $\boldsymbol{\gamma}$ is equivalent to the CCP property for (3.7) and follows from its CP property:

$$\left\langle\boldsymbol{\eta}^{k}|\boldsymbol{\gamma}\left(B_{k}^{*}B_{l}\right)\boldsymbol{\eta}^{l}\right\rangle=\left\langle\boldsymbol{\eta}^{k}|\boldsymbol{\varphi}\left(B_{k}^{*}B_{l}\right)\boldsymbol{\eta}^{l}\right\rangle\geq0$$

for such sequences $\boldsymbol{\eta}^{k}\in\mathcal{D}\oplus\mathcal{D}_{\bullet}$.

## 4. Construction of quantum CP flows

The necessary conditions for the stochastic generator $\boldsymbol{\lambda} = (\lambda^\mu_\nu)^{\mu=-,\bullet}_{\nu=+,\bullet}$ of a CP flow $\phi$ at $t=0$ are found in the previous section in the form of a CCP property for the corresponding germ $\boldsymbol{\gamma} = (\gamma^\mu_\nu)^{\mu=-,\bullet}_{\nu=+,\bullet}$. In the next section we shall show, these conditions are essentially equivalent to the assumption (3.8), corresponding to

$$\gamma^m(B) = \varphi^m(B) - K^*_m B = \gamma^*_m(B), \quad \gamma(B) = \varphi(B) - K^* B - BK, \tag{4.1}$$

where $\boldsymbol{\varphi} = (\varphi^\mu_\nu)^{\mu=-,\bullet}_{\nu=+,\bullet}$ is a CP map with $\varphi^m_n = \gamma^m_n$. Here we are going to prove under the following conditions for the operators $K, K_\bullet$ and the maps $\varphi^\mu_\nu$ that this general form is also sufficient for the existence of the CP solutions to the quantum stochastic equation (3.2). We are going to construct the minimal quantum stochastic positive flow $B \mapsto \phi_t(B)$ for a given w*-continuous unbounded germ-matrix map of the above form, satisfying the following conditions.

(1) First, we suppose that the operator $K \in \mathcal{B}(\mathcal{D})$ generates the one parametric semigroup $\left(e^{-Kt}\right)_{t>0}$, $e^{-Kr}e^{-Ks} = e^{-K(r+s)}$ of continuous operators $e^{-Kt} \in \mathcal{L}(\mathcal{D})$ in the strong sense

$$\lim_{t\searrow 0} \frac{1}{t}\left(I - e^{-Kt}\right)\eta = K\eta, \quad \forall \eta \in \mathcal{D}.$$

(A contraction semigroup on the Hilbert space $\mathcal{H}$ if $K$ defines an accretive $K + K^\dagger \geq 0$ and so maximal accretive form.)

(2) Second, we suppose that the solution $S^n_t, n \in \mathbb{N}$ to the recurrence

$$S^{n+1}_t = S^\circ_t - \int_0^t S^\circ_{t-r} \sum_{m=1}^\infty K_m S^n_r \mathrm{d}\Lambda^m_-, \quad S^0_t = S^\circ_t,$$

where $S^\circ_t = e^{-Kt} \otimes T_t \in \mathcal{L}(\mathfrak{D})$ is the contraction given by the shift coisometries $T_t : \mathfrak{F} \to \mathfrak{F}$, strongly converges to a continuous operator $S_t \in \mathcal{L}(\mathfrak{D})$ at $n \longrightarrow \infty$ for each $t > 0$.

(3) Third, we suppose that the solution $R^n_t, n \in \mathbb{N}$ to the recurrence

$$R^{n+1}_t = S^*_t S_t + \int_0^t \mathrm{d}\Lambda^\nu_\mu\left(r, S^*_r \varphi^\mu_\nu\left(R^n_{t-r}\right) S_r\right), \quad R^0_t = S^*_t S_t,$$

where the quantum stochastic non-adapted integral is understood in the sense [31], weakly converges to a continuous form $R_t \in \mathcal{B}(\mathfrak{D})$ at $n \longrightarrow \infty$ for each $t > 0$.

The first and second assumptions are necessary to define the existence of free evolution semigroup $S^\circ = (S^\circ_t)_{t>0}$ and its perturbation $S = (S_t)_{t>0}$ on the product space $\mathfrak{D} = \mathcal{D} \otimes \mathfrak{F}$ in the form of multiple quantum stochastic integral

$$S_t = S^\circ_t + \sum_{n=1}^\infty (-1)^n \int\!\!\cdots\!\!\int_{0<t_1<\ldots<t_n<t} K_{m_n}(t-t_n)\cdots K_{m_1}(t_2-t_1) S^\circ_{t_1} \mathrm{d}\Lambda^{m_1}_- \cdots \mathrm{d}\Lambda^{m_n}_-, \tag{4.2}$$

iterating the quantum stochastic integral equation

$$S_t = S^\circ_t - \int_0^t \sum_{m=1}^\infty K_m(t-r) S_r \mathrm{d}\Lambda^m_-, \quad S_0 = I, \tag{4.3}$$

where $K_m(t) = S_t^\circ (K_m \otimes I)$. The third assumption supplies the weak convergence for the series

(4.4)

$$R_t = S_t^* S_t + \sum_{n=1}^{\infty} \int \cdots \int_{0<t_1<\ldots<t_n<t} \mathrm{d}\Lambda^{\nu_1\ldots\nu_n}_{\mu_1\ldots\mu_n}\left(t_1,\ldots,t_n,\varphi^{\mu_1\cdots\mu_n}_{\nu_1\ldots\nu_n}\left(t_1,\ldots,t_n,S^*_{t-t_n}S_{t-t_n}\right)\right)$$

of non-adapted n-tuple integrals, i.e. for the multiple quantum stochastic integral [31] with

$$\varphi^{\mu_1\cdots\mu_n}_{\nu_1\ldots\nu_n}(t_1,\ldots,t_n) = \varphi^{\mu_1\ldots\mu_{n-1}}_{\nu_1\ldots\nu_{n-1}}(t_1,\ldots,t_{n-1}) \circ \varphi^{\mu_n}_{\nu_n}(t_n - t_{n-1}), \tag{4.5}$$

where $\varphi^\mu_\nu(t,B) = S_t^* \varphi^\mu_\nu(B) S_t$.

The following theorem gives a characterization of the evolution semigroup $S$ in terms of cocycles with unbounded coefficients, characterized by Fagnola [32] in the isometric and unitary case.

**Proposition 7.** *Let the family $V^\circ = (V_t^\circ)_{t>0}$ be a quantum stochastic adapted cocycle, $V_r^\circ T_s V_s^\circ = T_s V_{r+s}^\circ$, satisfying the HP differential equation*

$$\mathrm{d}V_t^\circ + KV_t^\circ \mathrm{d}t + \sum_{m=1}^{\infty} K_m V_t^\circ \mathrm{d}\Lambda^m_- + \sum_{n=1}^{\infty} V_t^\circ \mathrm{d}\Lambda^n_n = 0, \quad V_0^\circ = I. \tag{4.6}$$

*Then $S_t = T_t V_t^\circ$ is a semigroup solution, $S_r S_s = S_{r+s}$ to the non-adapted integral equation (4.3) such that $S_t \psi_f = S_t(f^\bullet)\eta \otimes \delta_\varnothing, \forall \eta \in \mathcal{D}$ on $\psi_f = \eta \otimes f^\otimes$ with $f^\bullet \in \mathfrak{E}_t$. Conversely, if $S = (S_t)_{t>0}$ is the non-adapted solution (4.2) to the integral equation (4.3), then $V_t^\circ = T_t^* S_t$ is the adapted solution to (4.6), defined as $V_t^\circ \psi_f = S_t(f^\bullet)\eta, \forall \eta \in \mathcal{D}$, where $S_t(f^\bullet) = F^* S_t F$ is given by $F\eta = \eta \otimes f^\otimes$ with $f^\bullet \in \mathfrak{E}_t$.*

*Proof.* First let us show that the equation (4.6) is equivalent to the integral one

$$V_t^\circ = e^{-Kt} \otimes I_t - \int_0^t \sum_{m=1}^{\infty} \left(e^{-K(t-r)} K_m \otimes I^r_{t-r}\right) V_r^\circ \mathrm{d}\Lambda^m_-, \quad V_0^\circ = I,$$

where $I_t = T_t^\dagger T_t = 0^{\Lambda^\bullet_\bullet(t)}$ is the decreasing family of orthoprojectors onto $\mathfrak{F}_{[t}$, and $I^r_s = \theta^r(I_s)$. Indeed, multiplying both parts of the integral equation from the left by $\left(e^{K(t-s)} \otimes I\right)$ and differentiating the product $e^{K(t-s)} V_t^\circ$ at $t = s$, we obtain (4.6) by taking into account that $\mathrm{d}I_t + \sum_{n=1}^{\infty} I_t \mathrm{d}\Lambda^n_n = 0$ and $\mathrm{d}\Lambda^n_n \mathrm{d}\Lambda^m_- = 0$. Conversely, the integral equation can be obtained from (4.6) by the integration:

$$\begin{aligned} V_t^\circ - e^{-Kt} \otimes I_t &= \int_0^t \mathrm{d}\left(\left(e^{-K(t-r)} \otimes I^r_{t-r}\right) V_r^\circ\right) \\ &= \int_0^t \left(e^{-K(t-r)} \otimes I^r_{t-r}\right)\left(\mathrm{d}V_r^\circ + KV_r^\circ \mathrm{d}r + V_r^\circ \mathrm{d}\Lambda^\bullet_\bullet\right) \\ &= -\int_0^t e^{-K(t-r)}\left(K_\bullet \otimes I^r_{t-r}\right) V_r^\circ \mathrm{d}\Lambda^\bullet_-, \end{aligned}$$

where we used that $\mathrm{d}I(r) = I(r)\,\mathrm{d}\Lambda^\bullet_\bullet$ and $\mathrm{d}\left(I(r)V_r^\circ\right) = \mathrm{d}I(r)\,V_r^\circ + I(r)\,\mathrm{d}V_r^\circ$ for the backward-adapted process $I(r) = I^r_{t-r}, \forall r \le t$. The non-adapted equation (4.3) is obtained by applying the operator $T_t = T_{t-r}T_r$ to both parts of this integral equation and taking into account the commutativity of $e^{K(r-t)}K_m$ with $T_r$. Moreover,

due to the adaptiveness of $V_t^\circ$, $S_t\psi_f = T_t\left(E_tV_t^\circ\psi_f \otimes E_{[t}f^\otimes\right) = S_t\left(f^\bullet\right)\eta\otimes f_t^\otimes$, where $f_t^\otimes = T_tf^\otimes$, and $S_t\left(f^\bullet\right) = EV_t^\circ F$ is the solution to the equation

$$S_t\left(f^\bullet\right) = e^{-Kt} + \int_0^t e^{-K(t-r)}K_\bullet f^\bullet\left(r\right)S_r\left(f^\bullet\right)\mathrm{d}r, \qquad S_0\left(f^\bullet\right) = I.$$

Hence $S_tF = E^*S_t\left(f^\bullet\right)$ if $f^\bullet \in \mathfrak{E}_t$, and $F^*S_tF = S_t\left(f^\bullet\right)$ as $EF = I$. Since this equation is equivalent to the differential one

$$\frac{\mathrm{d}}{\mathrm{d}t}S_t\left(f^\bullet\right)\eta + \left(K_\bullet f^\bullet\left(t\right) + K\right)S_t\left(f^\bullet\right)\eta = 0, \quad S_0\left(f^\bullet\right)\eta = \eta, \qquad \forall\eta\in\mathcal{D}, \tag{4.7}$$

the function $t\mapsto S_t\left(f^\bullet\right),\ f^\bullet\in\mathfrak{E}$ is a strongly continuous cocycle,

$$S_r\left(f_s^\bullet\right)S_s\left(f^\bullet\right) = S_{r+s}\left(f^\bullet\right),\ \forall r,s>0, \qquad f_s^\bullet\left(t\right) = f^\bullet\left(t+s\right), \quad S_0\left(f^\bullet\right) = I.$$

As was proved in [31], the multiple integral (4.2) gives a solution to the integral equation (4.3), and so the multiple integral for $V_t^\circ\psi_f = S_t\left(f^\bullet\right)\eta\otimes f^\otimes$,

$$S_t\left(f^\bullet\right) = e^{-Kt} + \sum_{n=1}^\infty\left(-1\right)^n \int\cdots\int_{0<t_1<\ldots<t_n<t} K\left(t,t_n\right)\cdots K\left(t_2,t_1\right)e^{-Kt_1}\mathrm{d}t_1\cdots\mathrm{d}t_n,$$

where $K\left(t,r\right) = e^{-K(t-r)}K_\bullet f^\bullet\left(r\right)$, corresponding to the iteration of the integral equation for $V_t^\circ$ on $\psi_f$, satisfies the HP equation (4.6). ∎

The following theorem reduces the problem of solving of differential evolution equations to the problem of iteration of integral equations similar to the nonstochastic case [33, 34].

**Proposition 8.** *Let $S_t = T_tV_t^\circ$, where $V_t^\circ\in\mathcal{L}\left(\mathfrak{D}\right)$ are continuous operators defining the adapted cocycle solution to the equation (4.6). Then the linear stochastic evolution equation (3.2) is equivalent to the quantum non-adapted (in the sense of* [31]*) integral equation*

$$\phi_t\left(B\right) = S_t^*BS_t + \int_0^t \mathrm{d}\Lambda_\mu^\nu\left(r,\phi_r\left[\varphi_\nu^\mu\left(S_{t-r}^*BS_{t-r}\right)\right]\right) \tag{4.8}$$

*with $\phi_0\left(B\right) = B\in\mathcal{B}$, where $\varphi_\nu^\mu$ are extended onto $\mathfrak{B}$ in the normal way by $w^*$-continuity and linearity as $\varphi_\nu^\mu\left(B\otimes Z\right) = \overline{\varphi}_\nu^\mu\left(B\right)\otimes Z$ for $B\in\overline{\mathcal{B}}$, $Z\in\mathcal{B}\left(\mathfrak{F}\right)$.*

The proof is given in [35, 10].

**Theorem 9.** *Let $\boldsymbol{\varphi}$ be a $w^*$-continuous CP-map, and $S_t = T_tV_t^\circ$ be given by the solution to the quantum stochastic equation (4.6). Then the solutions to the evolution equation (3.2) with the generators, corresponding to (4.1), have the CP property, and satisfy the submartingale (contractivity) condition $\phi_t\left(I\right)\le\epsilon_t\left[\phi_s\left(I\right)\right]$ for all $t<s$ if $\varphi\left(I\right)\le K+K^\dagger$ ($\phi_t(I)\le\phi_s(I)$ if $\boldsymbol{\varphi}(I)\le\boldsymbol{K}+\boldsymbol{K}^\dagger$). The minimal solution can be constructed in the form of multiple quantum stochastic integral in the sense* [31] *as the series*

$$\phi_t\left(B\right) = \sum_{n=0}^\infty \int\cdots\int_{0<t_1<\ldots<t_n<t} \mathrm{d}\Lambda_{\mu_1\ldots\mu_n}^{\nu_1\ldots\nu_n}\left(t_1,\ldots,t_n,\varphi_{\nu_1\ldots\nu_n}^{\mu_1\ldots\mu_n}\left(t_1,\ldots,t_n,S_{t-t_n}^*BS_{t-t_n}\right)\right) \tag{4.9}$$

*of non-adapted $n$-tuple CP integrals with $S_t^*BS_t$ at $n=0$ and*

$$\varphi_{\nu_1\ldots\nu_n}^{\mu_1\ldots\mu_n}\left(t_1,\ldots,t_n\right) = \varphi_{\nu_1}^{\mu_1}\left(t_1\right)\circ\varphi_{\nu_2}^{\mu_2}\left(t_2-t_1\right)\circ\ldots\circ\varphi_{\nu_n}^{\mu_n}\left(t_n-t_{n-1}\right),$$

*where* $\varphi_\nu^\mu(t,B) = S_t^* \varphi_\nu^\mu(B) S_t$. *If* $\boldsymbol{\varphi}$ *is bounded, then the solution to the equation is unique, and* $\phi_t(I) = \epsilon_t[\phi_s(I)]$ *for all* $t < s$ *if* $K + K^\dagger = \varphi(I)$ ($\phi_t(I) = I$ *if* $\boldsymbol{K} + \boldsymbol{K}^\dagger = \boldsymbol{\varphi}(I)$).

The proof is given in [35, 10].

## 5. The structure of the generators and flows

First, let us prove the structure (3.8) for the (unbounded) form-generator of CP flows over the algebra $\mathcal{B} = \mathcal{L}(\mathcal{H})$ of all bounded operators, assuming that $\mathcal{A} = 0$. This algebra contains the one-dimensional operators $|\eta'\rangle\langle\eta^0| : \eta \mapsto \langle\eta^0|\eta\rangle\,\eta'$ given by the vectors $\eta^0, \eta' \in \mathcal{H}$.

Let us fix a vector $\boldsymbol{\eta}^0 \in \mathcal{D} \oplus \mathcal{D}_\bullet$ with the unit projection $\eta^0 \in \mathcal{D}$, $\|\eta^0\| = 1$, and make the following assumption of the weak continuity for the linear operator $\eta' \mapsto \boldsymbol{\gamma}\left(|\eta'\rangle\langle\eta^0|\right)\boldsymbol{\eta}^0$.

0) The sequence $\boldsymbol{\eta}'_n = \boldsymbol{\gamma}\left(|\eta'_n\rangle\langle\eta^0|\right)\boldsymbol{\eta}^0 \in \mathcal{D}' \oplus \mathcal{D}'_\bullet$ of anti-linear forms

$$\boldsymbol{\eta} \in \mathcal{D} \oplus \mathcal{D}_\bullet \mapsto \langle\boldsymbol{\eta}|\boldsymbol{\eta}'_n\rangle := \left\langle\boldsymbol{\eta}|\boldsymbol{\gamma}\left(|\eta'_n\rangle\langle\eta^0|\right)\boldsymbol{\eta}^0\right\rangle$$

converges for each sequence $\eta'_n \in \mathcal{H}$ converging in $\mathcal{D}' \supseteq \mathcal{H}$.

**Proposition 10.** *Let the CCP germ-matrix* $\boldsymbol{\gamma}$ *satisfy the above continuity condition for a given* $\boldsymbol{\eta}^0$. *Then there exist strongly continuous operators* $K \in \mathcal{L}(\mathcal{D}), K_\bullet : \mathcal{D}_\bullet \to \mathcal{D}$ *defining the matrix operator* $\boldsymbol{K}$ *in (3.8), such that the matrix map (3.7) is CP, and there exists a Hilbert space* $\mathcal{K}$, *a* $*$*-representation* $\jmath : B \mapsto B \otimes J$ *of* $\mathcal{B} = \mathcal{L}(\mathcal{H})$ *on the Hilbert product* $\mathcal{G} = \mathcal{H} \otimes \mathcal{K}$, *given by an orthoprojector* $J$ *in* $\mathcal{K}$, *such that*

$$\boldsymbol{\varphi}(B) = \left(L^\mu \jmath(B) L_\nu\right)_{\nu=+,\bullet}^{\mu=-,\bullet} = \boldsymbol{L}^* \jmath(B) \boldsymbol{L}. \tag{5.1}$$

*Here* $\boldsymbol{L} = (L, L_\bullet)$ *is a strongly continuous operator* $\mathcal{D} \oplus \mathcal{D}_\bullet \to \mathcal{G}$ *with* $L = L_+$, $L^- = L^*$, $L^\bullet = L_\bullet^*$ *which is always possible to make*

$$\left\langle\eta^0 \otimes e|\boldsymbol{L}\boldsymbol{\eta}^0\right\rangle = 0, \qquad \forall e \in \mathcal{K}_1, \tag{5.2}$$

*where* $\mathcal{K}_1 = J\mathcal{K}$. *If* $D = -\lambda(I) \geq 0$, *then one can make* $L^*L = K + K^\dagger$ *in a canonical way* $L = L^\circ$, *and in addition one can make* $L^*L_\bullet = K_\bullet$, $L_\bullet^* L_\bullet = I_\bullet^\bullet$, *where* $I_\bullet^\bullet = I\delta_\bullet^\bullet$ *for a canonical* $L_\bullet = L_\bullet^\circ$ *if* $\boldsymbol{D} = -\boldsymbol{\lambda}(I) \geq 0$.

The proof is given in [10].

Thus we have proved that the equation (3.2) for a completely positive quantum stochastic flows over $\mathcal{B} = \mathcal{L}(\mathcal{H})$ has the following general form

$$\mathrm{d}\phi_t(B) + \phi_t\left(K^*B + BK - L^*\jmath(B)L\right)\mathrm{d}t = \sum_{m,n=1}^{\infty} \phi_t\left(L_m^*\jmath(B)L_n - B\delta_n^m\right)\mathrm{d}\Lambda_m^n$$

$$+ \sum_{m=1}^{\infty} \phi_t\left(L_m^*\jmath(B)L - K_m^*B\right)\mathrm{d}\Lambda_m^+ + \sum_{n=1}^{\infty} \phi_t\left(L^*\jmath(B)L_n - BK_n\right)\mathrm{d}\Lambda_-^n,$$

generalizing the Lindblad form [1] for the semigroups of completely positive maps. This can be written in the tensor notation form as

$$\mathrm{d}\phi_t(B) = \phi_t\left(L_\alpha^{\star\mu}\jmath_\beta^\alpha(B)L_\nu^\beta - \imath_\nu^\mu(B)\right)\mathrm{d}\Lambda_\mu^\nu = \phi_t\left(\mathbf{L}^\star \mathbf{\jmath}(B)\mathbf{L} - \mathbf{\imath}(B)\right)\cdot \mathrm{d}\mathbf{\Lambda}, \tag{5.3}$$

where the summation is taken over all $\alpha, \beta = -, \circ, +$ and $\mu, \nu = -, \bullet, +$, $\jmath_-^-(B) = B = \jmath_+^+(B)$, $\jmath_\circ^\circ(B) = \jmath(B)$, $\jmath_\beta^\alpha(B) = 0$ if $\alpha \neq \beta$, $\imath_\nu^\mu(B) = B\delta_\nu^\mu$, and $\mathbf{L}^\star = \left[L_\beta^{\star\mu}\right]_{\beta=-,\circ,+}^{\mu=-,\bullet,+}$

with $L^{\star\mu}_{-\alpha} = L^{\alpha*}_{-\mu}$ is the triangular matrix, pseudoadjoint to $\mathbf{L} = [L^{\alpha}_{\nu}]^{\alpha=-,\circ,+}_{\nu=-,\bullet,+}$ with $L^{-}_{-} = I = L^{+}_{+}$,

$$L^{\circ}_{\bullet} = L_{\bullet}, \qquad L^{\circ}_{+} = L, \qquad L^{-}_{\bullet} = -K_{\bullet}, \qquad L^{-}_{+} = -K.$$

(All other $L^{\alpha}_{\nu}$ are zero.) If the Hilbert space $\mathcal{K}$ is separable, $\mathcal{K}_1 = \ell^2(\mathbb{N}_1)$ for a subset $\mathbb{N}_1 \subseteq \mathbb{N}$. Then the equation (5.3) can be resolved as $\phi_t(B) = V^*_t(B \otimes I_t) V_t$, where $V = (V_t)_{t>0}$ is an (unbounded) cocycle on the product $\mathcal{D}\otimes\mathfrak{F}$ with Fock space $\mathfrak{F}$ over the Hilbert space $L^2(\mathbb{N}\times\mathbb{R}_+)$ of the quantum noise, and $I_t$ is a decreasing family of orthoprojectors in $\mathfrak{F}$, satisfying the stochastic equation $\mathrm{d}I_t + \sum_{n\notin\mathbb{N}_1} I_t \mathrm{d}\Lambda^n_n = 0$ with $I_0 = I$. The cocycle $V$ can be found from the quantum stochastic equation $\mathrm{d}V_t = (L^{\mu}_{\nu} - I\delta^{\mu}_{\nu})V_t \mathrm{d}\Lambda^{\nu}_{\mu}$ with $V_0 = I\otimes I$ of the form

(5.4)

$$\mathrm{d}V_t + KV_t\mathrm{d}t + \sum_{n=1}^{\infty} K_n V_t \mathrm{d}\Lambda^n_- = \sum_{m,n=1}^{\infty}(L^m_n - I\delta^m_n)V_t\mathrm{d}\Lambda^n_m + \sum_{m=1}^{\infty} L^m V_t \mathrm{d}\Lambda^+_m,$$

where $L^m_n$ and $L^m$ are the operators in $\mathcal{D}$, defining

$$\begin{aligned}(5.5)\qquad \varphi^m_n(B) &= \sum_{k\in\mathbb{N}_1} L^{k*}_m B L^k_n, \qquad \varphi(B) = \sum_{k\in\mathbb{N}_1} L^{k*} B L^k \\ \varphi^m(B) &= \sum_{k\in\mathbb{N}_1} L^{k*}_m B L^k, \qquad \varphi_n(B) = \sum_{k\in\mathbb{N}_1} L^{k*} B L^k_n\end{aligned}$$

with $\sum_{k=1}^{\infty} L^{k*}L^k = K + K^{\dagger}$ if $D \geq 0$, and in addition $\sum_{k=1}^{\infty} L^{k*}L^k_n = K_n$, $\sum_{k=1}^{\infty} L^{k*}_m L^k_n = I\delta^m_n$ if $\boldsymbol{D} \geq 0$. The formal derivation of the equation (5.4) from (5.3) is obtained by a simple application of the HP Itô formula. The martingale $M_t$, describing the density operator for the output state of $\Lambda(t,a)$, is then defined as $M_t = V^*_t V_t$.

The following theorem ensures the existence of a $*$-representation $\iota : \Lambda(t,a) \mapsto \Lambda(t, i(a)) := i^{\alpha}_{\beta}(a)\Lambda^{\beta}_{\alpha}(t)$ of the quantum stochastic process (0.2), commuting with $Y_t = \phi_t(B)$ for all $a \in \mathfrak{a}, B \in \mathcal{L}(\mathcal{H})$, with $A = X(0) = 0$, in the form

$$\Lambda(t, i(a)) = i^{\circ}_{\circ}(a)\Lambda^{\circ}_{\circ}(t) + i^{\circ}_{+}(a)\Lambda^{+}_{\circ}(t) + i^{-}_{\circ}(a)\Lambda^{\circ}_{-}(t) + i^{-}_{+}(a)\Lambda^{+}_{-}(t).$$

Here $\boldsymbol{i} = \left(i^{\alpha}_{\beta}\right)^{\alpha=-,\circ}_{\beta=+,\circ}$ is a $\star$-representation

$$i^{\alpha}_{\beta}(a^{\star}a) = i^{\alpha}_{\circ}(a^{\star})\, i^{\circ}_{\beta}(a), \quad i^{\alpha}_{-\beta}(a^{\star}) = i^{\beta}_{-\alpha}(a)^*$$

of the Itô algebra $\mathfrak{a}$ in the operators $i^{\alpha}_{\beta}(a) : \mathcal{K}_{\beta} \to \mathcal{K}_{\alpha}$, with a domain $\mathcal{K}_{\circ} \subseteq \mathcal{K}$, $\mathcal{K}_- = \mathbb{C} = \mathcal{K}_+$, and $\Lambda^{\beta}_{\alpha}(t)$ are the canonical quantum stochastic integrators in the Fock space $\Gamma(\mathfrak{K})$ over $\mathfrak{K} = L^2_{\mathcal{K}}(\mathbb{R}_+)$, the space of $\mathcal{K}$-valued square-integrable functions on $\mathbb{R}_+$.We shall extend $\boldsymbol{i}$ to the triangular matrix representation $\mathbf{i} = \left[i^{\alpha}_{\beta}\right]^{\alpha=-,\circ,+}_{\beta=-,\circ,+}$ on the pseudo-Hilbert space $\mathbb{C}\oplus\mathcal{K}\oplus\mathbb{C}$ with the Minkowski metrics tensor $\mathbf{g} = \left[\delta^{\alpha}_{-\beta}\right] = \mathbf{g}^{-1}$, by $i^{+}_{\beta}(a) = 0 = i^{\alpha}_{-}(a)$, for all $a \in \mathfrak{a}$, as it was done for $\mathbf{a} = [a^{\mu}_{\nu}]^{\mu=-,\bullet,+}_{\nu=-,\bullet,+}$, and denote the ampliation $I \otimes i^{\alpha}_{\beta}(a)$ again as $i^{\alpha}_{\beta}(a)$. Note that if the stochastic generator of the form (3.8) is restricted onto an operator algebra $\mathcal{B} \subseteq \mathcal{L}(\mathcal{H})$ with the weak closure $\bar{\mathcal{B}} = \mathcal{A}^c$, and all the sesquilinear forms $\gamma^{\mu}_{\nu}(B)$, $B \in \mathcal{B}$ commute with the $*$-algebra $\mathcal{A} \subset \mathcal{L}(\mathcal{D})$, then $\lambda^{\mu}_{\nu}(B) \in \bar{\mathcal{B}}$.

**Proposition 11.** *Let $\boldsymbol{b} = \boldsymbol{\gamma}(B) - \boldsymbol{\imath}(B)$ satisfy the commutativity conditions (3.4) for all $a \in \mathfrak{a}$, $B \in \mathcal{L}(\mathcal{H})$. Then there exists a $\star$-representation $a \mapsto \boldsymbol{i}(a)$ of the Itô*

*algebra* $\mathfrak{a}$*, defining the operators* $i^{\alpha}_{\beta}(a) : \mathcal{K}_{\beta} \to \mathcal{K}_{\alpha}$*, with* $i^{\alpha}_{\beta}(a)^{*}\mathcal{K}_{\alpha} \subseteq \mathcal{K}_{\beta}$*, where* $\mathcal{K}_{-} = \mathbb{C} = \mathcal{K}_{+}$*, such that* $L^{\alpha}_{\mu}(I \otimes a^{\mu}_{\nu}) = \left(I \otimes i^{\alpha}_{\beta}(a)\right) L^{\beta}_{\nu}$ *for all* $a \in \mathfrak{a}$*. By omitting* $I\otimes$ *this can be written as*

$$
\begin{aligned}
L_{\bullet}a^{\bullet}_{\bullet} &= i(a)\,L_{\bullet}, \qquad a^{-}_{+} - K_{\bullet}a^{\bullet}_{+} = i^{-}(a)\,L + i^{-}_{+}(a)\,,\\
L_{\bullet}a^{\bullet}_{+} &= i(a)\,L + i_{+}(a)\,, \qquad a^{-}_{\bullet} - K_{\bullet}a^{\bullet}_{\bullet} = i^{-}(a)\,L_{\bullet},
\end{aligned}
\tag{5.6}
$$

*where we take the convention* $i^{-} = i^{-}_{\circ}$*,* $i_{+} = i^{\circ}_{+}$ *and* $i = i^{\circ}_{\circ}$*. If* $\left[A, \gamma^{\mu}_{\nu}(B)\right] = 0$ *for all* $A \in \mathcal{A}$ *and* $B \in \mathcal{B}$*, where* $\mathcal{B} \subseteq \mathcal{L}(\mathcal{H})$ *is a* $*$*-subalgebra, and* $\vec{\mathcal{B}} = \mathcal{A}^{c}$*, then there exists a triangular* $\star$*-representation* $\mathbf{j} = \left[j^{\alpha}_{\beta}\right]^{\alpha=-,\circ,+}_{\beta=-,\circ,+}$ *of the operator algebra* $\mathcal{A}$ *with* $j^{\circ}_{\circ}(I) = J$ *such that*

$$
\mathbf{JLA} = \mathbf{j}(A)\,\mathbf{L}, \quad [\mathbf{j}(A)\,, \mathbf{i}(a)] = 0, \quad [\mathbf{j}(A)\,, \jmath(B)] = 0, \quad \forall A \in \mathcal{A}, a \in \mathfrak{a}, B \in \mathcal{B}.
\tag{5.7}
$$

The proof is given in [10] even for the general (noncommutative) Itô algebra $\mathfrak{a}$.

Now we are going to construct the quantum stochastic dilation for the flow $\phi_t(B)$ and the quantum state generating function $\vartheta^{a}_{t} = \epsilon\left[R_t W(t, a)\right]$ of the output process $\mathbf{\Lambda}(t, a)$ in the form

$$
\phi_t(B) = V^{*}_{t}(I_t \otimes B)\,V_t, \quad \vartheta_t(g) = \epsilon\left[V^{*}_{t}(W^{a}_{t} \otimes I)\,V_t\right], \quad \forall B \in \mathcal{L}(\mathcal{H})\,, a \in \mathfrak{a},
$$

where $V_t$ is an operator on $\mathfrak{D}$ into $\Gamma(\mathfrak{K}) \otimes \mathfrak{D}$, intertwining the Weyl operators $W(t, a)$ with the operators $W^{a}_{t} = W(t, i(a))\,I_t$ in the Fock space $\Gamma(\mathfrak{K})$,

$$
\mathrm{d}W(t, i(a)) = W(t, i(a))\,\mathrm{d}\Lambda(t, i(a))\,, \quad W(0, i(a)) = I,
$$

and $I_t \geq I_s, \forall t \leq s$ is a decreasing family of orthoprojectors.

In order to prove the existence of the Fock space dilation, we need the following assumptions in addition to the continuity assumptions of this and previous sections.

1) The minimal quantum stochastic $*$-flow [2] over the operator algebra $\mathcal{A}$, resolving the quantum Langevin equation

$$
\mathrm{d}\tau_t(A) = \tau_t(\mathbf{j}(A) - \imath(A)) \cdot \mathrm{d}\mathbf{\Lambda}, \quad A \in \mathcal{A},
$$

where $\mathbf{j}(I) = \mathbf{J} \otimes I$, $\imath(A) = \mathbf{I} \otimes A$ , constructed by its iterations with $\tau_0(A) = I \otimes A$ as it was done in the Sec 4 for the flow $\phi$, is the multiplicative flow, satisfying the condition $\tau_t(I) = I_t \otimes I$, where $I_t$ is the solution to $\mathrm{d}I_t = (J - I)^{\circ}_{\circ} I_t \mathrm{d}\Lambda^{\circ}_{\circ}$ with $I_0 = I$.

2) The operators $L_{\nu}(\bar{e}) = (I \otimes e^{*})\,L_{\nu}$, given for all $e \in \mathcal{G}$ as $\langle L_{\nu}(\bar{e})\,\eta|\eta'\rangle = \langle L_{\nu}\eta|\eta' \otimes e\rangle \quad \forall \eta \in \mathcal{D}, \eta' \in \mathcal{D}'$, are strongly continuous onto $\mathcal{D}$. This is necessary for the weak definition of the operators $V_t(\sigma) : \mathfrak{D} \to \mathcal{K}^{\otimes|\sigma|} \otimes \mathfrak{D}_{[t}$ on finite subsets $\sigma \subset [0, t)$ by the recurrence

$$
V_t(\sigma)\,\psi = \left(I^{\otimes|\sigma|} \otimes V^{\circ}_{t}(s)\right)\left(LV_s(\sigma\backslash s)\,\psi + \sum_{n=1}^{\infty} L_n V_s(\sigma\backslash s)\,\psi^{n}(s)\right),
$$

with $V_t(\emptyset) = V^{\circ}_{t}(0)\,, \quad s = \max\sigma$. Here $V^{\circ}_{t}(s) = T^{*}_{s}V^{\circ}_{t-s}T_s$, $V^{\circ}_{t}$ is the solution to the equation (4.3), the operators $L_{\nu} : \mathcal{D} \to \mathcal{K}\otimes\mathcal{D}$ act on $\mathcal{K}^{\otimes|\sigma\backslash s|}\otimes\mathfrak{D}_{[s}$ as $I^{\otimes|\sigma\backslash s|} \otimes L_{\nu} \otimes I_{[s}$, and $\psi^{n}(s)$ are the components of $\psi^{\bullet}(\tau, s) = \psi(\tau \sqcup s)$, where $\tau \sqcup s$ is defined for almost all $s$ ($s \notin \sigma$) as the disjoint union of the single point $\{s\}$ with a finite subset $\tau \in \mathbb{R}_{+}$.

3) The operator-valued function $\sigma \mapsto V_t(\sigma)$, defined for all such $\sigma \in \Gamma_t$, is weakly square integrable for each $t$ with respect to the measure $\mathrm{d}\sigma = \prod_{s\in\sigma} \mathrm{d}s$ in the sense

$$\int_{\Gamma_t} \|V_t(\sigma)\psi\|^2 \mathrm{d}\sigma := \sum_{n=0}^{\infty} \int \dots \int_{0<s_1 \dots s_n<t} \|V_t(s_{1,\dots,}s_n)\psi\|^2 \mathrm{d}s_1 \dots \mathrm{d}s_1 < \infty,$$

for all $\psi \in \mathfrak{D}$. Thus the operators $V_t(\cdot)$ define a Fock space one $V_t : \mathfrak{D} \to \Gamma(\mathfrak{K}_t) \otimes \mathfrak{D}_{[t}$. They form a cocycle, $V^r_{t-r}(\sigma) V_r(\sigma) = V_t(\sigma)$, where $V^r_s(\sigma) = I^{\otimes|\sigma_r|} \otimes T^*_r V_s(\sigma_{[r} - r) T_r$.

**Theorem 12.** *Under the given assumptions 0)–3) there exist:*

(i) *A cocycle dilation $V_t : \mathfrak{D} \to \Gamma(\mathfrak{K}_t) \otimes \mathfrak{D}$ of the minimal CP flow $\phi$, intertwining the Weyl operator $W(t,a)$ with $W^a_t$:*

$$V_t(I \otimes W(t,a)) = (W^a_t \otimes I) V_t, \quad \phi_t(B) = V^*_t(I_t \otimes B) V_t\ , \quad \forall a \in \mathfrak{a}, B \in \mathcal{L}(\mathcal{H}), \tag{5.8}$$

*where $I_t \le I_s$, $\forall t < s$ are orthoprojectors in $\Gamma(\mathfrak{K})$;*

(ii) *A $*$-multiplicative flow $\tau = (\tau_t)$ over $\mathcal{A}$ in $\Gamma(\mathfrak{K}) \otimes \mathcal{H}$ with the properties $\tau_t(I) = I_t$,*

$$V_t A = \tau_t(A) V_t, \quad [\tau_t(A), W^a_t] = 0, \quad [\tau_t(A), I \otimes B] = 0, \quad \forall A \in \mathcal{A}, a \in \mathfrak{a}, B \in \mathcal{B}. \tag{5.9}$$

(iii) *If $\lambda(I) \le 0$, then one can make $M_t = V^*_t V_t$ martingale, and, if $\boldsymbol{\lambda}(I) \le 0$, one can make $V_t$ isometric, $V^*_t V_t = I$.*

(iv) *Moreover, let $U = (U_t)_{t\ge0}$ be a one-parametric weakly continuous cocycle of unitary operators on $\Gamma(\mathfrak{K}) \otimes \mathcal{H} \otimes \Gamma(\mathfrak{E})$ , satisfying the quantum stochastic equation*

$$\begin{aligned} &\mathrm{d}U_t + \left(K\mathrm{d}t + K^-_\bullet \mathrm{d}\Lambda^\bullet_- + K^-_\circ \mathrm{d}\Lambda^\circ_-\right) U_t \\ = \;& \left(L^\circ_+ \mathrm{d}\Lambda^+_\circ - I^\bullet_\bullet \mathrm{d}\Lambda^\bullet_\bullet + J^\circ_\bullet \mathrm{d}\Lambda^\bullet_\circ + J^\bullet_\circ \mathrm{d}\Lambda^\circ_\bullet + (J^\circ_\circ - I^\circ_\circ)\mathrm{d}\Lambda^\circ_\circ\right) U_t \end{aligned} \tag{5.10}$$

*with $U_0 = I$ and the necessary differential unitarity conditions*

$$K + K^\dagger = L^-_\circ L^\circ_+,\ K^-_\bullet = L^-_\circ J^\circ_\bullet,\ J^\bullet_\circ J^\circ_\bullet = I^\bullet_\bullet,\ K^-_\circ = L^-_\circ J^\circ_\circ,\ J^\circ_\circ = I^\circ_\circ - J^\circ_\bullet J^\bullet_\circ,$$

*where $L^-_\circ = L^{\circ*}_+$, $J^\bullet_\circ = J^{\circ*}_\bullet$. If $\lambda(I) \le 0$ and $L^\circ_+ = L^\circ$ is the canonical operator in the dilation (5.1), then*

$$\langle\psi|(A \otimes I)\phi^a_t(B)\psi\rangle = \langle U_t(\delta_\emptyset \otimes \psi)\,|\,(\tau^a_t(A)(I \otimes B)) U_t(\delta_\emptyset \otimes \psi)\rangle \tag{5.11}$$

*for all $A \in \mathcal{A}, a \in \mathfrak{a}, B \in \mathcal{B}$ and any initial $\psi = \eta \otimes \delta_\emptyset$, $\eta \in \mathcal{D}$, where*

$$\phi^a_t(B) = (I \otimes W(t,a))\phi_t(B), \quad \tau^a_t(A) = (W^a_t \otimes I)\tau_t(A).$$

*This unitary cocycle dilation is valied for any state $\psi \in \mathcal{D} \otimes \mathfrak{F}$ if in addition $J^\circ_\bullet = L^\circ_\bullet$ is the canonical isometry in (5.1) for the case $\boldsymbol{\lambda}(I) \le 0$.*

*Proof.* (Sketch). The cocycle $V = (V_t)_{t>0}$ is recurrently constructed due to the above assumptions (1)–(3). It obviously intertwines the Weyl operators (2.4) with the operators $W^a_t$, acting in the same way in $\Gamma(\mathfrak{K})$, by virtue of the property (5.6).

Let us denote by $\mathfrak{K}_1 = L^2_{\mathcal{K}}(\mathbb{R}_+)$ the functional Hilbert space corresponding to the minimal dilation (5.1) sub-space $\mathcal{K} = \mathcal{K}_1$ for the CP map $\boldsymbol{\varphi}$, given by the

orthoprojector $J = J_1$ in the space $\mathcal{K}_\circ$ of the canonical dilation, and $\mathfrak{K}_0$ its orthogonal compliment, corresponding to $\mathcal{K}_0 = J_0 \mathcal{K}_\circ$, where $J_0 = I - J_1$. Representing $\Gamma(\mathfrak{K}_0 \oplus \mathfrak{K}_1)$ as $\Gamma(\mathfrak{K}_0) \otimes \Gamma(\mathfrak{K}_1)$, let us denote by $I_t$ the survival orthoprojectors

$$I_t \chi\left(\sigma^0, \sigma^1\right) = \delta_\emptyset\left(\sigma_t^0\right) \chi\left(\sigma^0, \sigma^1\right), \qquad \sigma_t = \sigma \cap [0, t),$$

where $\chi\left(\sigma^0, \sigma^1\right) = \chi\left(\sigma^0 \sqcup \sigma^1\right) \in \mathcal{K}^{\otimes|\sigma^0|} \otimes \mathcal{K}^{\otimes|\sigma^1|}$ is the set function, representing a $\chi \in \Gamma(\mathfrak{K}_0 \oplus \mathfrak{K}_1)$. The decreasing family $(I_t)_{t>0}$ defines the decay orthoprojectors $E_t = I - I_t$ in $\Gamma(\mathfrak{K}_\circ)$ satisfying the quantum stochastic equation $\mathrm{d}E_t = E_t J_0 \cdot \mathrm{d}\Lambda_\circ^\circ$ with $E_0 = 0$, and $\Lambda_\circ^\circ$ is the number integrator in the Fock space $\Gamma(\mathfrak{K}_\circ)$ over $\mathfrak{K}_\circ = \mathfrak{K}_0 \oplus \mathfrak{K}_1$. Then one easily find that the minimal CP flow (4.9) can be represented as $\phi_t(B) = V_t^*(I_t \otimes B) V_t$.

We may also construct by iteration the minimal quantum stochastic $*$-flow $\tau = (\tau_t)$ over the operator algebra $\mathcal{A}$, resolving the quantum Langevin equation by its iteration. It is unique if normaliesed $\tau_t(I) = I_t$ to the solution of the Langevin equation for $A = I$, and it is $*$-multiplicative as in [30] due to the differential $*$-multiplicativity of $\mathbf{j}$. Then the properties (5.9) follow from the definition of the operators $\widehat{V}_t$, and can be checked recurrently by use of (5.6) and (5.7).

The cocycle $U = (U_t)$ is constructed to give the unitary solution to the HP equation (1.10), which always exists due to differential unitarity relations. If the solution is unique as in the case of all bounded coefficients $K$ and $L$, it can be represented in the form of the stochastic multiple integral of the chronologically ordered products of the coefficients of the quantum differential equation under the integrability conditions given in [10].

If $K + K^\dagger \geq \varphi(I)$, the HP unitarity condition [6] is satisfied for the canonical choice $L_+^\circ = L^\circ$ and arbitrary isometric operator $J_\bullet^\circ$, $J_\circ^\bullet J_\bullet^\circ = I_\bullet^\bullet$ with $K_\bullet^- = L_\circ J_\bullet^\circ$, $K_\circ^- = L_\circ J_\circ^\circ$, $J_\circ^\circ = I_\circ^\circ - J_\bullet^\circ J_\circ^\bullet$; if $\boldsymbol{K} + \boldsymbol{K}^\dagger \geq \boldsymbol{\varphi}(I)$, in addition we make the choice $J_\bullet^\circ = L_\bullet^\circ$ from the canonical dilation, and so $K_\bullet^- = L_\circ L_\bullet^\circ = K_\bullet$, where $L_\circ^* = L^\circ$, $J_\circ^\bullet = J_\bullet^{\circ *}$. In the first, subfiltering case $\lambda(I) \leq 0$ such a choice gives the coincidence $U_t(\delta_\emptyset \otimes \psi_0) = V_t \psi_0$ of the stochastic multiple integrals for any initial vacuum $\psi_0 = \eta \otimes \delta_\emptyset$, $\eta \in \mathcal{D}$, and therefore $\|V_t \psi_0\| = \|\psi_0\|$. Thus $M_t = V_t^* V_t$ is a martingale, and the equation (5.11) is satisfied for any initial $\psi_0$. In the second, contractive case $\boldsymbol{\lambda}(I) \leq 0$ the canonical choice gives $U_t(\delta_\emptyset \otimes \psi) = V_t \psi$ and therefore $\|V_t \psi\| = \|\psi\|$ for any $\psi \in \mathcal{D} \otimes \mathfrak{F}$. Thus $V_t^* V_t = I$, and the equation (5.11) is satisfied for any state $\psi$. ∎

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